

# ADVANCED MATHEMATICS

TRIGONOMETRY

# 7

$$\sec^2(X) - \tan^2(X) = 1$$

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# TRIGONOMETRY

Chapter

# 7

The word **trigonometric** is derived from two Greek words **trigon**, which means triangles, and **metric**, which means measures. Knowledge of the trigonometrical ratios is vital in very many fields of engineering, mathematics and physics.

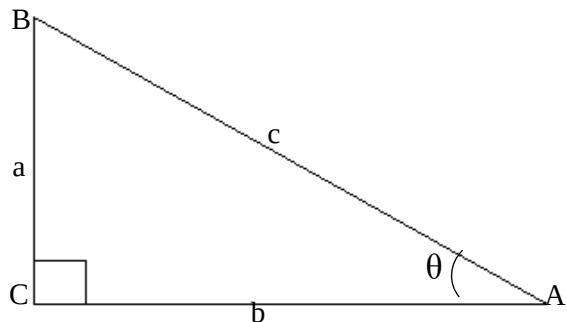
Trigonometry, as the word implies, is concerned with the measurement of the parts, sides and angles, of a triangle. The early applications of the trigonometric functions were to surveying, navigation, and engineering. These functions also play an important role in the study of all sorts of vibratory phenomena-sound, light, electricity, etc. Therefore, a considerable portion of the subject matter is concerned with a study of the properties of and relations among the trigonometric functions.

## Trigonometric ratios as ratios of a right-angled triangle

$$\sin \theta = \frac{\text{opposite}}{\text{Hypotenuse}} = \frac{a}{c}$$

$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{b}{c}$$

$$\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{a}{b} = \frac{\sin \theta}{\cos \theta}$$



### Reciprocal of trigonometric ratios

Reciprocal of sine is cosecant,  $\text{cosecant} = \frac{1}{\sin}$  which is abbreviated as cosec or csc

Reciprocal of cosine is secant,  $\text{secant} = \frac{1}{\cos}$ , which is abbreviated as sec.

Reciprocal of tangent is cotangent,  $\text{cotangent} = \frac{1}{\tan}$ , this is abbreviated as cot

$$\text{Therefore, } \begin{cases} \sec x = \frac{1}{\cos x} \\ \text{cosec } x = \frac{1}{\sin x} \\ \cot x = \frac{1}{\tan x} \end{cases}$$

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### Trigonometric identities

Consider the triangle  $ABC$  given

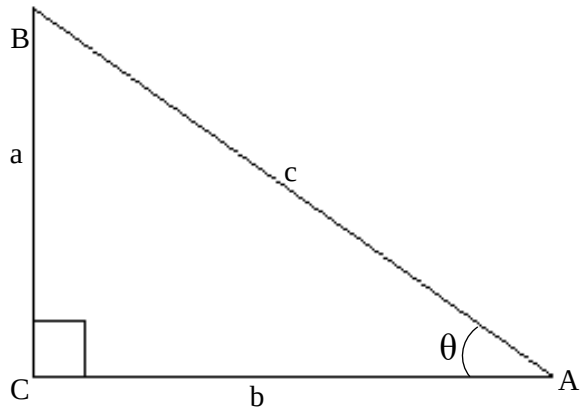
$$\sin \theta = \frac{a}{c} \text{ and } \cos \theta = \frac{b}{c}$$

$$\sin^2 \theta + \cos^2 \theta = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$

$$\sin^2 \theta + \cos^2 \theta = \frac{a^2 + b^2}{c^2}$$

By Pythagoras theorem

$$a^2 + b^2 = c^2$$



$$\sin^2 \theta + \cos^2 \theta = \frac{c^2}{c^2} = 1, \text{ therefore } \sin^2 \theta + \cos^2 \theta = 1$$

Using  $\sin^2 \theta + \cos^2 \theta = 1$  divide throughout by  $\cos^2 \theta$

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \text{ therefore } 1 + \tan^2 \theta = \sec^2 \theta$$

Using  $\sin^2 \theta + \cos^2 \theta = 1$  divide throughout by  $\sin^2 \theta$

$$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \text{ therefore } \cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$$

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### SIMPLIFYING TRIGONOMETRIC IDENTITIES

This means to write the compound trigonometric given in a most simplified form with one or two trigonometric functions.

#### Example 1

Simplify  $\frac{\tan^2 \theta + 1}{\tan \theta \operatorname{cosec}^2 \theta}$

#### Solution

Given  $\frac{\tan^2 \theta + 1}{\tan \theta \operatorname{cosec}^2 \theta}$  from  $1 + \tan^2 \theta = \sec^2 \theta$  then  $\frac{\sec^2 \theta}{\left(\frac{\sin \theta}{\cos \theta}\right)\left(\frac{1}{\sin^2 \theta}\right)}$

$$= \frac{\sec^2 \theta}{1} = \frac{1}{\cos^2 \theta} \times (\cos \theta \sin \theta)$$

$$= \frac{\sin \theta}{\cos \theta}$$

Therefore,  $\frac{\tan^2 \theta + 1}{\tan \theta \operatorname{cosec}^2 \theta} = \tan \theta$

### Example 2

Simplify  $\operatorname{cosec} x \cos^2 x + \sin x$

#### Solution

Given  $\operatorname{cosec} x \cos^2 x + \sin x$

$$\left( \frac{1}{\sin x} \right) \cos^2 x + \sin x$$

$$\frac{\cos^2 x + \sin^2 x}{\sin x} = \frac{1}{\sin x} = \operatorname{cosec} x$$

Therefore,  $\operatorname{cosec} x \cos^2 x + \sin x = \operatorname{cosec} x$

### Example 3

Simplify  $\frac{1}{\sec x - 1} + \frac{1}{\sec x + 1}$

#### Solution

Given  $\frac{1}{\sec x - 1} + \frac{1}{\sec x + 1}$

$$= \frac{(\sec x + 1) + (\sec x - 1)}{(\sec x - 1)(\sec x + 1)} = \frac{2 \sec x}{\sec^2 x - 1} = \frac{2 \sec x}{\tan^2 x}$$

$$= \left( \frac{2}{\cos x} \right) \left( \frac{\cos^2 x}{\sin^2 x} \right) = \frac{2 \cos x}{\sin^2 x} = 2 \operatorname{cosec} x \cot x$$

Therefore,  $\frac{1}{\sec x - 1} + \frac{1}{\sec x + 1} = 2 \operatorname{cosec} x \cot x$

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## EXERCISE 1

(a) Simplify the following trigonometric identities

1.  $\frac{\tan x + \sin x}{\csc x}$

2.  $\frac{\csc \theta - \sin \theta}{\cot^2 \theta}$

3.  $\cot x(\cos x \tan x + \sin x)$

4.  $\cos \theta + \sin \theta \tan \theta$

5.  $\sin^2 x - \cos^2 x \sin^2 x$

6.  $\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x}$

7.  $\frac{\sec x + \csc x}{1 + \tan x}$

8.  $(1 - \sin x)(\sec x + \tan x)$

9.  $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

10.  $(\sec x - \tan x)(\csc x + 1)$

11.  $(1 + \tan x)^2 + (1 - \tan x)^2$

12.  $(\cos x + 3 \sin x)^2 + (3 \cos x - \sin x)^2$

13.  $\sqrt{(3 \cos x + 4 \sin x)^2 + (4 \cos x - 3 \sin x)^2}$

14.  $\tan^2 x - \sin^2 x$

15.  $\frac{\csc^2 x - \cot^2 x}{\tan^2 x}$

16.  $\frac{\sin x \cos x}{1 - \cos^2 x}$

17.  $\tan x - \sin x \cos x$

18.  $\frac{1}{\sin^2 x} - \frac{1}{\tan^2 x}$

19.  $\sin x(\csc x - \sin x)$

20.  $\frac{\cos x}{\sec x} + \frac{\sin x}{\csc x}$

(b) From each of equation set given eliminate  $\theta$ 

21.  $x = 3 \sin \theta$  and  $y = \operatorname{cosec} \theta$

22.  $x = 3 + \sin \theta$  and  $y = \tan \theta$

23.  $5x = \sin \theta$  and  $y = 2 \cos \theta$

24.  $x = 2 \sin \theta$  and  $3y = \cos \theta$

25.  $x = 3 + \sin \theta$  and  $y = \operatorname{cosec} \theta$

26.  $x = \cot \theta$  and  $y = a + \sec \theta$

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### PROOF OF TRIGONOMETRIC IDENTITIES

The first part above was to simplify trigonometric identities, in this part, we are required to verify that the right-hand side is equals to the left hand side. We are required to simplify one side to be the same as the other side. To verify, we can start with either side.

#### Example 4

Show that  $\sec x - \sin x \tan x = \cos x$

#### Solution

$$\sec x - \sin x \tan x = \cos x$$

Consider **left hand side**

$$\frac{1}{\cos x} - \sin x \left( \frac{\sin x}{\cos x} \right) = \frac{1 - \sin^2 x}{\cos x} = \frac{\cos^2 x}{\cos x} = \cos x, \text{ hence LHs become as}$$

RHs, hence shown.

#### Example 5

Prove that  $\tan x + \cot x = \sec x \operatorname{cosec} x$

#### Solution

Consider left hand side

$$\tan x + \cot x$$

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x} = \csc x \sec x \text{ hence shown.}$$

#### Example 6

Prove that  $\frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$

#### Solution

Consider left hand side

$\frac{\sin x}{1 + \cos x} \times \frac{1 - \cos x}{1 - \cos x}$  (Multiplying by  $1 - \cos x$  in numerator and denominator)

$$\frac{\sin x(1 - \cos x)}{1 - \cos^2 x} = \frac{\sin x(1 - \cos x)}{\sin^2 x} = \frac{1 - \cos x}{\sin x} \text{ hence shown}$$

### Example 7

Prove that  $\cot x - \tan x = \frac{2 \cos^2 x - 1}{\sin x \cos x}$

### Solution

Consider the left hand side

$$\cot x - \tan x$$

$$\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}$$

$$\frac{\cos^2 x - (1 - \cos^2 x)}{\sin x \cos x} = \frac{2 \cos^2 x - 1}{\sin x \cos x}$$

## EXERCISE 2

Prove each of the following identities

1.  $\csc x \cos^2 x + \tan x = \csc x$
2.  $\frac{\sin \theta + \tan \theta}{1 + \cos \theta} = \tan \theta$
3.  $\frac{\cot \theta - \tan \theta}{\cos \theta + \sin \theta} = \frac{\cos \theta - \sin \theta}{\sin \theta \cos \theta}$
4.  $\frac{1 - \sin \theta}{\cos \theta} = \frac{1}{\sec \theta + \tan \theta}$
5.  $\frac{\tan \theta - \cot \theta}{\tan \theta + \cot \theta} = 1 - 2 \cos^2 \theta$
6.  $\frac{\csc x + 1}{\csc x - 1} = (\sec x + \tan x)^2$
7.  $(\csc x - 1)(\sin x + 1) = (\cos x + \tan x)^2$
8.  $\sin^2 x + \cot^2 x + \sec^2 x = \csc^2 x$
9.  $\sin^2 x(1 + \tan^2 x) = \sec^2 x - 1$
10.  $\cos^2 x - \sin^2 x = 2 \cos^2 x - 1$

11.  $\sec^2 x + \csc^2 x = \frac{\sec^2 x}{\sin^2 x}$
12.  $\sec x - \cos x = \sin x \tan x$
13.  $\csc \theta - \sin \theta = \cos \theta \cot \theta$
14.  $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$
15.  $\tan^2 \theta + \tan \theta \cot \theta = \sec^2 \theta$
16.  $\frac{1}{1 - \sin x} - \frac{1}{1 + \sin x} = 2 \tan x \sec x$
17.  $\frac{1 - \tan \theta}{1 + \tan \theta} = \frac{\cot \theta - 1}{\cot \theta + 1}$
18.  $\frac{\sec x - 1}{\tan x} = \frac{\tan x}{\sec x + 1}$
19.  $\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \csc x - \cot x$
20.  $\frac{\cos \theta}{\sec \theta + \tan \theta} = 1 - \sin \theta$
21.  $\frac{\sec \theta - 1}{\sin^2 \theta} = \frac{\sec^2 \theta}{1 + \sec \theta}$
22.  $(a \cos \theta + b \sin \theta)^2 + (b \cos \theta + a \sin \theta)^2 = a^2 + b^2$
23.  $\frac{\sin^2 x + 2 \cos x - 1}{\sin^2 x + 3 \cos x - 3} = \frac{1}{1 - \sec x}$
24. If  $x = 2 + \sin \theta$  and  $y + 1 = \cos \theta$  show that  $x^2 + y^2 - 4x + 2y + 4 = 0$

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### TRIGONOMETRIC EQUATION

Trigonometric equations may have infinity solutions; to be specific the range of the solution should be given. The range may be  $0^\circ \leq \theta \leq 360^\circ$ ,  $-360^\circ \leq \theta \leq 360^\circ$  or  $-180^\circ \leq \theta \leq 180^\circ$  etc. if not given we give out the general solution. Note that, this range may also be given in radians. We shall also refer the four quadrants of the  $xy$ -plane to get the required principal solutions.

#### Example 8

Solve for  $x$  from  $0^\circ \leq x \leq 360^\circ$ ;  $\sin(x + 15^\circ) = -0.3$

#### Solution

$$\text{Given } \sin(x + 15^\circ) = -0.3$$

$$x + 15^\circ = \sin^{-1}(0.3)$$

$$x + 15^\circ = -17.46^\circ$$

$$x + 15^\circ = 197.46^\circ, 342.54^\circ$$

$$x = 182.46^\circ, 327.54^\circ$$

**Example 9**

$$\text{Solve } \sin(x + 10^\circ) = -0.5 \text{ from } 0^\circ \leq x \leq 360^\circ$$

**Solution**

$$\sin(x + 10^\circ) = -0.5$$

$$x + 10^\circ = -\sin^{-1}(0.5)$$

$$x + 10^\circ = -30^\circ \text{ then } x + 10^\circ = 210^\circ, 330^\circ$$

$$\text{Therefore } x = 200^\circ, 320^\circ$$

**Example 10**

$$\text{Solve } \cos x = -0.4 \text{ from } -180^\circ \leq x \leq 180^\circ$$

**Solution**

$$\cos x = -0.4$$

$$x = \cos^{-1}(-0.4)$$

$$x = 113.58^\circ, \text{ or } x = -113.58^\circ$$

**Example 11**

$$\text{Solve } 3\tan\theta + 1 = 0 \text{ from } -180^\circ \leq \theta \leq 180^\circ$$

**Solution**

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$$\text{Given } 3 \tan \theta + 1 = 0$$

$$\tan \theta = -\frac{1}{3}$$

$$\theta = \tan^{-1}\left(-\frac{1}{3}\right)$$

$$\theta = -18.43^\circ \text{ or } \theta = 161.57^\circ$$

**Example 12**

$$\text{Solve } \sin^2 x + \sin x \cos x = 0 \text{ from } 0^\circ \leq x \leq 360^\circ$$

**Solution**

$$\sin^2 x + \sin x \cos x = 0$$

$$\sin x (\sin x + \cos x) = 0$$

$$\text{Case 1: } \sin x = 0$$

$$x = \sin^{-1}(0)$$

$$x = 0^\circ, 180^\circ \text{ or } 360^\circ$$

$$\text{Case 2: } \sin x + \cos x = 0$$

$$\sin x = -\cos x \text{ therefore } \tan x = -1$$

$$x = \tan^{-1}(-1)$$

$$x = 135^\circ \text{ or } 315^\circ$$

$$\text{Therefore, } x = 0^\circ, 135^\circ, 180^\circ, 315^\circ \text{ or } 360^\circ$$

**Example 13**

$$\text{Solve } 6\cos^2 x - \sin x - 1 = 0 \text{ from } 0^\circ \leq x \leq 360^\circ$$

**Solution**

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$$6\cos^2 x - \sin x - 1 = 0$$

$$6(1 - \sin^2 x) - \sin x - 1 = 0$$

$$6 - 6\sin^2 x - \sin x - 1 = 0$$

$$-6\sin^2 x - \sin x + 5 = 0$$

$$(6\sin x - 5)(\sin x + 1) = 0$$

$$\text{Case 1: } 6\sin x - 5 = 0$$

$$\sin x = \frac{5}{6} \Rightarrow x = \sin^{-1}\left(\frac{5}{6}\right)$$

$$x = 56.44^\circ \text{ or } 123.56^\circ$$

$$\text{Case 2: } \sin x = -1$$

$$x = \sin^{-1}(-1)$$

$$x = 270^\circ$$

Therefore,  $x = 56.44^\circ, 123.56^\circ$  or  $270^\circ$

#### Example 14

Solve  $4\sin x \cos x = 3\cos x$  for  $0^\circ \leq x \leq 360^\circ$

#### Solution

$$4\sin x \cos x = 3\cos x$$

$$4\sin x \cos x - 3\cos x = 0$$

$$\sin x(4\cos x - 3) = 0$$

$$\text{Case 1: } \sin x = 0$$

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$$x = \sin^{-1}(0)$$

$$x = 0^\circ, 180^\circ \text{ or } 360^\circ$$

$$\text{Case 2: } 4\cos x - 3 = 0$$

$$\cos x = \frac{3}{4}$$

$$x = \cos^{-1}\left(\frac{3}{4}\right)$$

$$x = 48.59^\circ \text{ or } 131.41^\circ$$

The solutions are  $x = 0^\circ, 48.59^\circ, 131.41^\circ, 180^\circ \text{ or } 360^\circ$

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### Example 15

Solve  $2 \tan \theta - \frac{1}{\tan \theta} = 1$  from  $0^\circ \leq \theta \leq 360^\circ$

#### Solution

$$2 \tan \theta - \frac{1}{\tan \theta} = 1$$

$$2 \tan^2 \theta - 1 = \tan \theta$$

$$2 \tan^2 \theta - \tan \theta - 1 = 0$$

$$(\tan \theta - 1)(2 \tan \theta + 1) = 0$$

$$\text{Case 1: } \tan \theta = 1$$

$$\theta = \tan^{-1}(1)$$

$$\theta = 45^\circ \text{ or } 225^\circ$$

$$\text{Case 2: } \tan \theta = -\frac{1}{2}$$

$$\theta = \tan^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = 153.44^\circ \text{ or } 206.56^\circ$$

The solution are  $\theta = 45^\circ, 153.44^\circ, 225^\circ, 206.56^\circ$  or  $225^\circ$

### Example 16

Solve  $3\sin^2 \theta = 2\sin \theta \cos \theta$  from  $0^\circ \leq \theta \leq 360^\circ$

#### Solution

$$3\sin^2 \theta = 2\sin \theta \cos \theta$$

$$3\sin^2 \theta - 2\sin \theta \cos \theta = 0$$

$$\sin \theta (3\sin \theta - 2\cos \theta) = 0$$

$$\text{Case 1: } \sin \theta = 0$$

$$\theta = 0^\circ, 180^\circ \text{ or } 360^\circ$$

$$\text{Case 2: } 3\sin \theta = 2\cos \theta$$

$$\tan \theta = \frac{2}{3}$$

$$\theta = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\theta = 33.69^\circ \text{ or } 146.31^\circ$$

The solutions are  $\theta = 0^\circ, 33.69^\circ, 146.31^\circ, 180^\circ$  or  $360^\circ$

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## EXERCISE 3

(a) Solve the following from  $0^\circ$  to  $360^\circ$  inclusive

1.  $\cos 2x = 0.7$
2.  $2 \tan 3x - 1 = 0$
3.  $\cos^2 \theta - \sin \theta \cos \theta = 0$
4.  $5 \sin^2 \theta - 2 \sin \theta - 1 = 0$
5.  $2 \sin \theta - 3 \cos \theta = 0$
6.  $5 \cos \theta + 3 = 0$
7.  $\cos \theta (2 \sin \theta + \cos \theta) = 0$
8.  $4 \sin x \cos x = 3 \sin x$
9.  $(2 \sin \theta - 1)(\sin \theta + 1) = 0$
10.  $\sin \theta - 2 \sin \theta \cos \theta = 0$
11.  $\tan(x + 30^\circ) = 2$
12.  $4 \cos^2 \theta = \cos \theta \csc \theta$
13.  $\tan^2 \theta + 3 \sec^2 \theta = 7$
14.  $2 \sin^2 x - 5 \sin x + 2 = 0$
15.  $2 \tan x + 1 = 8 \tan^2 x$

(b) Solve the following from  $-180^\circ$  to  $180^\circ$  inclusive

16.  $\sin 2x = \cos 3x$
17.  $2 \sin^2 x - 4 \sin x + 1 = 0$
18.  $3 \sin^2 x = 2 \sin x \cos x$
19.  $\cos(x - 30^\circ) = \frac{1}{\sqrt{2}}$
20.  $4 \cos x - 3 \sec x = 2 \tan x$
21.  $\sin^2 x + \cos x + 1 = 0$
22.  $4 - \sin x = 4 \cos^2 x$
23.  $5 - 5 \cos x = 3 \sin^2 x$
24.  $\cos^2 x + 7 \cos x = 3$
25.  $8 \tan x = 3 \cos x \sin x$

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## COMPOUND ANGLES, NATURAL SINE $\sin(A \pm B)$

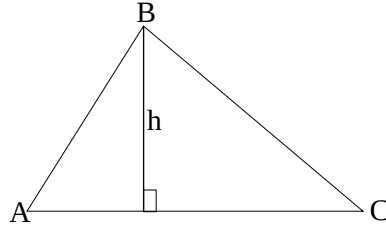
Before we proceed with the compound angles, we need first to recall the sine rule

Consider the triangle below

Area of triangle ABC,  $A = \frac{1}{2}bh$

$$\sin A = \frac{\text{opp}}{\text{Hyp}} = \frac{h}{c} \Rightarrow h = c \sin A$$

$$A = \frac{1}{2}bc \sin A$$



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Using this area formula, we can deduce the sine rule as follows

$$A = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$$

Simplifying

$$\frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$$

Divide throughout by  $abc$ ,

$$ab \sin C = bc \sin A = ac \sin B$$

$$\frac{ab \sin C}{abc} = \frac{bc \sin A}{abc} = \frac{ac \sin B}{abc}$$

This is sine rule

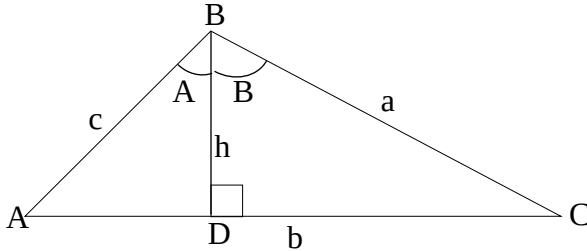
$$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

Sine rule is used to solve triangle problem of the two cases below

- (a) Solving ASA (Angle, Side, Angle) and AAS (Angle, Angle, Side) Cases
- (b) Solving the Ambiguous SSA (Side, Side, Angle) Case

Using the above sine rule

Consider again the triangle ABC below



Area of triangle ABC = Area of triangle ADB + Area of triangle DBC

$$\text{Area of ABC, } A = \frac{1}{2}ac\sin(A+B)$$

$$\text{Area of triangle ADB, } A = \frac{1}{2}hc\sin A \text{ but } \cos A = \frac{h}{c} \Rightarrow h = c \cos A$$

$$\text{Area of triangle DBC, } A = \frac{1}{2}ah\sin B \text{ but } \cos B = \frac{h}{a} \Rightarrow h = a \cos B$$

Area of triangle ABC = Area of triangle ADB + Area of triangle DBC

$$A = \frac{1}{2}ac\sin(A+B) = \frac{1}{2}hc\sin A + \frac{1}{2}ah\sin B$$

$$ac\sin(A+B) = hc\sin A + ah\sin B$$

$$ac\sin(A+B) = (a \cos B)(c \sin A) + (a \sin B)(c \cos A)$$

$$ac\sin(A+B) = ac\sin A \cos B + accos A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Therefore,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

Sine and Tangent are odd functions, thus  $\sin(-A) = -\sin A$  and  $\tan(-A) = -\tan A$  while cosine is an even function, thus  $\cos(-A) = \cos A$ , this apply to secant, cosecant and cotangent.

Using the concept of odd function of sine, we can proof  $\sin(A - B)$

Replace,  $B$  with  $-B$ ,

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

Therefore,

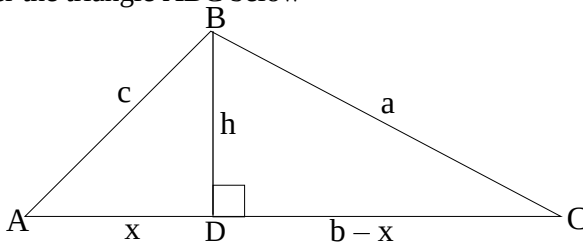
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

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### COMPOUND ANGLES, NATURAL COSINE $\cos(A \pm B)$

Before showing these compound angles, we shall first derive the cosine formula, which will be our base in proving the cosine compound angles

Consider the triangle ABC below



Using Pythagoras theorem, in triangle DBC

$$(b - x)^2 + h^2 = a^2$$

$$b^2 - 2bx + x^2 + h^2 = a^2$$

$$\cos A = \frac{x}{c} \Rightarrow x = c \cos A$$

$$\sin A = \frac{h}{c} \Rightarrow h = c \sin A$$

$$b^2 - 2b(c \cos A) + (c \cos A)^2 + (c \sin A)^2 = a^2$$

$$b^2 - 2bcc \cos A + c^2 \cos^2 A + c^2 \sin^2 A = a^2$$

$$b^2 - 2bcc \cos A + c^2 (\cos^2 A + \sin^2 A) = a^2$$

$$a^2 = b^2 + c^2 - 2bcc \cos A$$

This is cosine rule,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

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Note that, cosine rule is used in two cases

- (a) Solving the SAS (Side, Angle, Side) Case
- (b) Solving the SSS (Side, Side, Side) Case

Now, consider the triangle ABC, let b – x be y

$$(x + y)^2 = a^2 + b^2 - 2ab \cos(A + B)$$

$$x^2 + 2xy + y^2 = a^2 + b^2 - 2ab \cos(A + B)$$

$$\text{In triangle ADC: } h^2 = b^2 - y^2$$

(Pythagoras theorem)

$$\text{In triangle DBC: } h^2 = a^2 - x^2$$

$$-(x^2 - a^2) + -(y^2 - b^2) - 2xy = -2ab \cos(A + B)$$

$$-h^2 - h^2 + 2xy = -2ab \cos(A + B)$$

$$-2h^2 + 2xy = -2ab\cos(A+B)$$

$$-h^2 + xy = -ab\cos(A+B)$$

$$ab\cos(A+B) = xy - h^2$$

$$\cos(A+B) = \frac{xy}{ab} - \frac{h^2}{ab}$$

$$\cos(A+B) = \left(\frac{x}{a}\right)\left(\frac{y}{b}\right) - \left(\frac{h}{a}\right)\left(\frac{h}{b}\right)$$

$$\text{But } \cos A = \frac{y}{b}, \cos B = \frac{x}{a}, \sin A = \frac{h}{b} \text{ and } \sin B = \frac{h}{a}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Therefore,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Using the concept of odd and even functions, we deduce the formula

By replacing  $B$  with  $-B$ ,

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

From the compound angles  $\sin(A \pm B)$  and  $\cos(A \pm B)$  we deduce the compound angles of  $\tan(A \pm B)$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

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$$\tan(A+B) = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$\tan(A+B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$\tan(A+B) = \frac{\left(\frac{\sin A}{\cos A}\right) + \left(\frac{\sin B}{\cos B}\right)}{1 - \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Therefore,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

For  $\tan(A-B)$

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$$

$$\tan(A-B) = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

$$\tan(A+B) = \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}}$$

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$$\tan(A-B) = \frac{\left(\frac{\sin A}{\cos A}\right) - \left(\frac{\sin B}{\cos B}\right)}{1 + \left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Therefore,

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

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## PROVING TRIGONOMETRIC IDENTITIES

### Example 17

Show that  $\cos(A+B)\cos(A-B) = \cos^2 A - \cos^2 B$

#### Solution

$$\cos(A+B)\cos(A-B) = \cos^2 A - \cos^2 B$$

$$(\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$$

$$\cos^2 A \cos^2 B - \sin^2 A \sin^2 B$$

$$\cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B$$

$$\cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B$$

$$\cos^2 A - \sin^2 B, \text{ hence proved}$$

### Example 18

Show that  $\sin(90^\circ - A) = \cos A$

**Solution**

$$\sin(90^\circ - A) = \cos A$$

$$\sin(90^\circ - A) = \sin 90^\circ \cos A - \cos 90^\circ \sin A$$

But  $\sin 90^\circ = 1$  and  $\cos 90^\circ = 0$

$$\sin(90^\circ - A) = (1)\cos A - (0)\sin A$$

$$\sin(90^\circ - A) = \cos A, \text{ Hence shown.}$$

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**Example 19**

Show that  $\frac{\sin(A+B)}{\sin(A-B)} = \frac{\tan A + \tan B}{\tan A - \tan B}$

**Solution**

Consider the left hand side

$$\frac{\sin(A+B)}{\sin(A-B)} = \frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B - \cos A \sin B}$$

$$\begin{aligned} & \frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B} \\ &= \frac{\cos A \cos B}{\sin A \cos B} - \frac{\cos A \cos B}{\cos A \sin B} \end{aligned}$$

$$\begin{aligned} & \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} \\ &= \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B} \end{aligned}$$

$$= \frac{\tan A + \tan B}{\tan A - \tan B} \text{ Hence shown}$$

### EXERCISE 4

(a) Prove the following identities

1.  $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

2.  $\frac{\cos(A+B)}{\cos(A-B)} = \frac{1 + \tan A \tan B}{1 - \tan A \tan B}$

3.  $\frac{\sin(A-B)}{\cos A \cos B} + \frac{\sin(B-C)}{\cos B \cos C} = \frac{\sin(A-C)}{\cos A \cos C}$

4.  $\cos(\pi - A) = -\cos A$

5.  $\sin\left(\frac{\pi}{2} + A\right) = -\cos A$

6. If  $A+B=C$ ,  $\tan A = p \tan B$ , show that  $\sin(A-B) = \frac{p-1}{p+1} \sin C$

7. If  $\sin x + \sin y = a$  and  $\cos x + \cos y = b$  show that

$$\cos(x-y) = \frac{a^2}{2} + \frac{b^2}{2} - 1$$

8. The roots of the equation  $8x^2 - 26x + 15 = 0$  are  $\tan A$  and  $\tan B$  find  
(a)  $\cos(A+B)$  and (b)  $\sin(A-B)$ .

(b) Prove the following

9.  $\frac{\tan(A+B) - \tan B}{1 + \tan(A+B)\tan B} = \tan A$

10.  $\frac{\sin(A+B)}{\cos(A-B)} = \frac{\cot A + \cot B}{1 + \cot A \cot B}$

11.  $\frac{\sin(A+B)}{\sin(A-B)} = \frac{\tan A + \tan B}{\tan A - \tan B}$

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**SIMPLIFYING TRIGONOMETRIC IDENTITIES****Example 20**

Simplify  $\frac{\sin(x-y)}{\cos x \cos y}$

**Solution**

$$\begin{aligned}\frac{\sin(x-y)}{\cos x \cos y} \\&= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} \\&= \frac{\sin x \cos y}{\cos x \cos y} - \frac{\cos x \sin y}{\cos x \cos y} \\&= \frac{\sin(x-y)}{\cos x \cos y} = \tan x - \tan y\end{aligned}$$

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**Example 21**

Simplify  $\frac{\sin(A-B)}{\sin B} + \frac{\cos(A-B)}{\cos B}$

**Solution**

$$\begin{aligned}\frac{\sin(A-B)}{\sin B} + \frac{\cos(A-B)}{\cos B} \\&= \frac{\sin A \cos B - \cos A \sin B}{\sin B} + \frac{\cos A \cos B + \sin A \sin B}{\cos B} \\&= \frac{\sin A \cos B}{\sin B} - \cos A + \cos A + \frac{\sin A \sin B}{\cos B}\end{aligned}$$

$$\begin{aligned}
&= \frac{\sin A \cos B}{\sin B} + \frac{\sin A \sin B}{\cos B} \\
&= \frac{\sin A \cos^2 B + \sin A \sin^2 B}{\sin B \cos B} \\
&= \frac{\sin A (\cos^2 B + \sin^2 B)}{\sin B \cos B} \\
&= \frac{\sin A}{\cos B \sin B}
\end{aligned}$$

### Example 22

Simplify  $\frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ}$

### Solution

$$\frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ}$$

Recall that  $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ} = \tan(30^\circ + 15^\circ)$$

$$\frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ} = \tan 45^\circ$$

$$\frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ} = 1$$

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**Example 23**

Simplify  $\tan(x-y) - \tan(y-x)$

**Solution**

Tangent is an odd function, this mean that  $\tan(-A) = -\tan(A)$

$$\tan(x-y) - \tan(y-x)$$

$$= \tan(x-y) - \tan(-(x-y))$$

$$= \tan(x-y) + \tan(x-y)$$

$$= 2 \tan(x-y)$$

$$= 2 \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

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**DOUBLE ANGLES FORMULAE****Deriving double angles from compound angles**

From  $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Let  $B = A$

$$\sin(A+A) = \sin A \cos A + \cos A \sin A$$

Therefore,  $\sin 2A = 2 \sin A \cos A$

From  $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Let  $B = A$

$$\cos(A+A) = \cos A \cos A - \sin A \sin A$$

Therefore,  $\cos 2A = \cos^2 A - \sin^2 A$

$$\text{From } \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

### HALF ANGLE FORMULA

From the double angle formulae, we deduce the half angles.

From  $\sin 2A = 2 \sin A \cos A$  replace  $A$  by  $\frac{A}{2}$

The half angle formula for sine,  $\sin A = 2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)$

From  $\cos 2A = \cos^2 A - \sin^2 A$

Half angle for cosine,  $\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$

From  $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$  replace  $A$  by  $\frac{A}{2}$

Half angle formula for tangent

$$\tan A = \frac{2 \tan\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)}$$

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**Example 24**

Prove that  $\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos 2A$

**Solution**

$$\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos 2A$$

$$= \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{\left(\cos^2 A - \sin^2 A\right) / \cos^2 A}{\left(\cos^2 A + \sin^2 A\right) / \cos^2 A}$$

$$= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

$$= \cos^2 A - \sin^2 A = \cos 2A, \text{ hence shown}$$

**Example 25**

Show that  $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2 \theta$

**Solution**

$$\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2 \theta$$

$$= \frac{1 - \cos^2 \theta + \sin^2 \theta}{1 + \cos^2 \theta - \sin^2 \theta}$$

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$$\begin{aligned} &= \frac{2\sin^2 \theta}{1 + \cos^2 \theta - 1 + \cos^2 \theta} \\ &= \frac{2\sin^2 \theta}{2\cos^2 \theta} = \tan^2 \theta \text{ shown} \end{aligned}$$

**Example 26**

Prove that  $\csc 2\theta + \cot 2\theta = \cot \theta$

**Solution**

$$\begin{aligned} \csc 2\theta + \cot 2\theta &= \cot \theta \\ &= \frac{1}{\sin 2\theta} + \frac{1}{\tan 2\theta} \\ &= \frac{1}{\sin 2\theta} + \frac{\cos 2\theta}{\sin 2\theta} \\ &= \frac{1 + \cos 2\theta}{\sin 2\theta} \\ &= \frac{1 + \cos^2 \theta - \sin^2 \theta}{2\sin \theta \cos \theta} \\ &= \frac{1 + \cos^2 \theta - (1 - \cos^2 \theta)}{2\cos \theta \sin \theta} \\ &= \frac{2\cos^2 \theta}{2\sin \theta \cos \theta} = \frac{\cos \theta}{\sin \theta} = \cot \theta \text{ Shown} \end{aligned}$$

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**Example 27**

Prove that  $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta$

**Solution**

$$\begin{aligned}\frac{\sin 2\theta}{1 + \cos 2\theta} &= \tan \theta \\&= \frac{2\sin \theta \cos \theta}{1 + \cos^2 \theta - \sin^2 \theta} \text{ thus } = \frac{2\sin \theta \cos \theta}{1 + \cos^2 \theta - (1 - \cos^2 \theta)} \\&= \frac{2\sin \theta \cos \theta}{2\cos^2 \theta} \\&= \frac{\sin \theta}{\cos \theta} = \tan \theta, \text{ Shown.}\end{aligned}$$

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**Example 28**

Show that  $\cos^2 x + \cos 2x = 2 - 3\sin^2 x$

**Solution**

$$\begin{aligned}\cos^2 x + \cos 2x &= 2 - 3\sin^2 x \\ \cos^2 x + \cos 2x &= \cos^2 x + \cos^2 x - \sin^2 x \\ &= 1 - \sin^2 x + 1 - \sin^2 x - \sin^2 x \\ &= 2 - 3\sin^2 x, \text{ Hence shown}\end{aligned}$$

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EXERCISE 5

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Prove the following identities

1.  $\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1 + \cos A}$

2.  $\sin \theta = \frac{2 \tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}$

3.  $2 \csc 2\theta - \cot \theta = \tan \theta$

4.  $\tan 2\theta - 2 \tan 2\theta \sin^2 \theta = \sin 2\theta$

5.  $\frac{\sin \theta + \sin 2\theta}{1 + \cos \theta + \cos 2\theta} = \tan \theta$

6.  $\cos 2\theta = \cos^4 \theta - \sin^4 \theta$

7.  $\cot \theta - 2 \cot 2\theta = \tan \theta$

8.  $\frac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$

9.  $\frac{\sin(x-y)}{\sin y} + \frac{\cos(x-y)}{\cos y} = \frac{2 \sin x}{\sin 2y}$

10.  $\frac{\cos(x-y)}{\cos x \cos y} = \tan x + \cot y$

11.  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

12.  $\cos 3\theta + 3 \cos \theta = 4 \cos^3 \theta$

13.  $(\sin A + \csc A)^2 = \sin^2 A + \cot^2 A + 3$

14.  $\sin 3\alpha = \sin \alpha (3 \cos^2 \alpha - \sin^2 \alpha)$

15.  $\cot \theta - \tan \theta = 2 \cot 2\theta$

16.  $\sec(A+B) = \frac{\sec A \sec B}{1 - \tan A \tan B}$

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17. If  $2A + B = 45^\circ$  prove that  $\tan B = \frac{1 - \tan^2 A - 2 \tan A}{1 - \tan^2 A + 2 \tan A}$

18. If  $\sin(x + y) = 2 \cos(x - y)$  show that  $\tan x = \frac{2 - \tan y}{1 - 2 \tan y}$

19.  $\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos \theta}{\sin \theta}$

20. If  $\tan \theta = \frac{5}{12}$  find (a)  $\tan 2\theta$  (b)  $\tan 3\theta$

21. If  $A + B = 45^\circ$  show that  $\tan A = \frac{\cos B - \sin B}{\cos B + \sin B}$

## SIMPLIFYING TRIGONOMETRIC IDENTITIES INVOLVING DOUBLE ANGLES

### Example 29

Simplify  $\sqrt{\frac{1 + \sin 2A}{1 - \sin 2A}}$

#### Solution

$$\begin{aligned} & \sqrt{\frac{1 + \sin 2A}{1 - \sin 2A}} \\ &= \sqrt{\frac{1 + \sin 2A}{1 - \sin 2A} \times \frac{1 + \sin 2A}{1 + \sin 2A}} \\ &= \sqrt{\frac{(1 + \sin 2A)^2}{1 - \sin^2 2A}} \\ &= \sqrt{\frac{(1 + \sin 2A)^2}{\cos^2 2A}} = \frac{1 + \sin 2A}{\cos 2A} \end{aligned}$$

$$\begin{aligned}
&= \frac{1 + 2\sin A \cos A}{\cos^2 A - \sin^2 A} \\
&= \frac{\cos^2 A + \sin^2 A + 2\sin A \cos A}{\cos^2 A - \sin^2 A} \\
&= \frac{(\cos A + \sin A)^2}{\cos^2 A - \sin^2 A} = \frac{(\cos A + \sin A)(\cos A + \sin A)}{(\cos A + \sin A)(\cos A - \sin A)} \\
&= \frac{\cos A + \sin A}{\cos A - \sin A} = \frac{1 + \tan A}{1 - \tan A}
\end{aligned}$$

**Example 30**

Simplify  $\frac{\sin 2x}{\sin x}$

**Solution**

$$\frac{\sin 2x}{\sin x}$$

$$\frac{2\sin x \cos x}{\sin x} = 2\cos x$$

**Example 31**

Simplify  $\sin(x-y)\cos y + \cos(x-y)\sin y$

**Solution**

$$\sin(x-y)\cos y + \cos(x-y)\sin y$$

Let  $x - y = A$

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$$= \sin A \cos y + \cos A \sin y$$

$$= \sin(A + y) \text{ but } x - y = A$$

$$= \sin(x - y + y)$$

$$= \sin x$$

### EXERCISE 6

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Simplify the following questions

1.  $\frac{\sin 2A}{\cos A}$
2.  $2 \tan A \cos^2 A$
3.  $2 \sin^2 A + \cos 2A$
4.  $3 \cos 2A + 4 \sin A$
5.  $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x}$
6.  $\frac{3 \sin x + \sin 2x}{1 + 3 \cos x + \cos 2x}$
7.  $\sin\left(\frac{\pi}{6} + A\right) - \sin\left(\frac{\pi}{6} - A\right) = \cos A$
8.  $\sqrt{\frac{1 - \cos A}{2}}$
9.  $\sqrt{\frac{1 + \cos A}{1 - \cos A}}$
10.  $\cos^2 A + \tan 2A$

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**SOLVING EQUATION INVOLVING DOUBLE ANGLE****Example 32**

Solve  $3\sin 2x + \cos x = 0$  from  $0^\circ \leq x \leq 360^\circ$

**Solution**

$$3\sin 2x + \cos x = 0$$

$$6\sin x \cos x + \cos x = 0$$

$$\cos x(6\sin x + 1) = 0$$

Case 1:  $\cos x = 0$

$$x = \cos^{-1}(0)$$

$$x = 90^\circ \text{ or } 270^\circ$$

Case 2:  $6\sin x + 1 = 0$

$$\sin x = -\frac{1}{6}$$

$$x = \sin^{-1}\left(-\frac{1}{6}\right)$$

$$x = 189.59^\circ \text{ or } 350.41^\circ$$

Therefore,  $x = 90^\circ, 189.59^\circ, 270^\circ$  or  $350.41^\circ$

**Example 33**

Solve  $\sin 2x - \sin x = 0$

**Solution**

$$\sin 2x - \sin x = 0$$

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$$2\sin x \cos x - \sin x = 0$$

$$\sin x(2\cos x - 1) = 0$$

$$\text{Case 1: } \sin x = 0$$

$$x = \sin^{-1}(0)$$

$$x = 0^\circ, 180^\circ, 360^\circ$$

$$\text{Case 2: } 2\cos x - 1 = 0$$

$$\cos x = 0.5$$

$$x = \cos^{-1}(0.5)$$

$$x = 60^\circ \text{ or } 300^\circ$$

The solution set is  $x = 0^\circ, 60^\circ, 180^\circ, 300^\circ$  and  $360^\circ$

### Example 34

Solve  $3\cot 2x + \cot x = 1$  from  $-180^\circ \leq x \leq 180^\circ$

#### Solution

$$3\cot 2x + \cot x = 1$$

$$3\left(\frac{1}{\tan 2x}\right) + \frac{1}{\tan x} = 1$$

$$\frac{3 - 3\tan^2 x}{2\tan x} + \frac{1}{\tan x} = 1$$

$$3 - 3\tan^2 x + 2 = 2\tan x$$

$$3\tan^2 x + 2\tan x - 5 = 0$$

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$$(3 \tan x + 5)(\tan x - 1) = 0$$

$$\text{Case 1: } 3 \tan x + 5 = 0$$

$$\tan x = -\frac{5}{3}$$

$$x = \tan^{-1}\left(-\frac{5}{3}\right)$$

$$x = -59.04^\circ, 120.96^\circ$$

$$\text{Case 2: } \tan x = 1$$

$$x = \tan^{-1}(1)$$

$$x = 45^\circ$$

The solution set is  $x = -59.04^\circ, 45^\circ, 120.96^\circ$

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### EXERCISE 7

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Solve the following from  $0^\circ \leq \theta \leq 360^\circ$

1.  $\cos 2\theta = \sin \theta + 3$
2.  $\cos 3\theta = \sin(\theta + 10^\circ)$
3.  $\sin 2\theta \cos \theta + \sin^2 \theta = 1$
4.  $3 \cos^2 2\theta + 2 \cos 2\theta = 1$
5.  $5 \cot^2 3\theta - \cot 3\theta = 2$
6.  $2 \tan^2 \theta = \tan 2\theta + 1$
7.  $\sin 2\theta = \cos 2\theta + 1$
8.  $\tan \theta = \cot \theta$
9.  $\sec^4 2\theta = 4$
10.  $3 \sin^2 3\theta - 5 \sin 3\theta + 2 = 0$

**SOLUTION OF EQUATIONS OF THE FORM  $a\cos\theta + b\sin\theta = c$** 

(a) From the half angle formula, we derive the so called  $t$  formula,

$$\sin A = 2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)$$

$$\sin A = \frac{2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)}{\cos^2\left(\frac{A}{2}\right) + \sin^2\left(\frac{A}{2}\right)}$$

$$\sin A = \frac{2 \tan\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$$

$$\text{Let } t = \tan\left(\frac{A}{2}\right)$$

$$\text{Therefore, } \sin A = \frac{2t}{1+t^2}$$

$$(b) \cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$$

$$\cos A = \frac{\cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)}{\cos^2\left(\frac{A}{2}\right) + \sin^2\left(\frac{A}{2}\right)}$$

$$\cos A = \frac{1 - \tan^2\left(\frac{A}{2}\right)}{1 + \tan^2\left(\frac{A}{2}\right)}$$

$$\text{Let } t = \tan\left(\frac{A}{2}\right)$$

$$\text{Therefore, } \cos A = \frac{1-t^2}{1+t^2}$$

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$$(c) \tan A = \frac{2 \tan\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)}$$

$$\text{Let } t = \tan\left(\frac{A}{2}\right)$$

$$\text{Then, } \tan A = \frac{2t}{1-t^2}$$

### Example 35

Solve  $2\cos\theta + \sin\theta - 1 = 0$  from  $0^\circ \leq \theta \leq 360^\circ$

**Solution**

$$2\cos\theta + \sin\theta - 1 = 0$$

$$2\left(\frac{1-t^2}{1+t^2}\right) + \left(\frac{2t}{1+t^2}\right) - 1 = 0$$

$$2 - 2t^2 + 2t - (1+t^2) = 0$$

$$3t^2 - 2t - 1 = 0$$

$$(t-1)(3t+1) = 0$$

$$t = 1 \text{ or } t = -\frac{1}{3}$$

$$\text{But } t = \tan\left(\frac{\theta}{2}\right)$$

$$\text{Case 1: } \tan\left(\frac{\theta}{2}\right) = 1$$

$$\frac{\theta}{2} = \tan^{-1}(1)$$

$$\frac{\theta}{2} = [45^\circ, 225^\circ]$$

$$\theta = 90^\circ, 450^\circ$$

$$\text{Case 2: } \tan\left(\frac{\theta}{2}\right) = -\frac{1}{3}$$

$$\frac{\theta}{2} = \tan^{-1}\left(-\frac{1}{3}\right)$$

$$\frac{\theta}{2} = [161.57^\circ, 341.57^\circ]$$

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$$\theta = 323.14^\circ, 683.14^\circ$$

The solution is  $\theta = 90^\circ, 323.14^\circ$

Note that, other solutions are out of range given.

### Example 36

Solve equation  $3\cos\theta + 4\sin\theta = 2$  from  $0^\circ \leq \theta \leq 360^\circ$

#### Solution

$$3\cos\theta + 4\sin\theta = 2$$

$$3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) = 2$$

$$3 - 3t^2 + 8t = 2 + 2t^2$$

$$5t^2 - 8t - 1 = 0$$

$$t = 1.7165 \text{ or } t = -0.1165$$

$$\tan\left(\frac{\theta}{2}\right) = 1.7165$$

$$\frac{\theta}{2} = \tan^{-1}(1.7165)$$

$$\frac{\theta}{2} = [59.77^\circ, 239.77^\circ]$$

$$\theta = 119.54^\circ, 479.54^\circ$$

$$\tan\left(\frac{\theta}{2}\right) = -0.1165$$

$$\frac{\theta}{2} = \tan^{-1}(-0.1165)$$

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$$\theta/2 = [173.36^\circ, 353.36^\circ]$$

$$\theta = 346.72^\circ, 706.72^\circ$$

The solution set is  $\theta = 119.54^\circ, 346.72^\circ$

### Example 37

Solve  $\sqrt{3}\cos\theta + \sin\theta = 1$  form  $0^\circ \leq \theta \leq 360^\circ$

#### Solution

$$\sqrt{3}\cos\theta + \sin\theta = 1$$

$$\sqrt{3}\left(\frac{1-t^2}{1+t^2}\right) + \left(\frac{2t}{1+t^2}\right) = 1$$

$$\sqrt{3}(1-t^2) + 2t = 1+t^2$$

$$\sqrt{3} - 1 - \sqrt{3}t^2 - t^2 + 2t = 0$$

$$-(\sqrt{3}+1)t^2 + 2t + \sqrt{3} - 1 = 0$$

$$t = 1 \text{ or } t = -0.2679$$

$$\tan(\theta/2) = 1$$

$$\theta/2 = \tan^{-1}(1)$$

$$\theta/2 = [45^\circ, 225^\circ]$$

$$\theta = 90^\circ, 450^\circ$$

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$$\tan\left(\frac{\theta}{2}\right) = -0.2679$$

$$\frac{\theta}{2} = \tan^{-1}(-0.2679)$$

$$\frac{\theta}{2} = [165^\circ, 345^\circ]$$

$$\theta = 330^\circ, 690^\circ$$

$$\therefore \theta = 90^\circ, 330^\circ$$

**Solution of equations of the form  $a \cos \theta + b \sin \theta = c$  :**

**(a) Expressing equation of the form  $a \cos \theta + b \sin \theta = c$  into the form**

$$a \cos \theta + b \sin \theta = R \cos(\theta - \alpha)$$

$$a \cos \theta + b \sin \theta = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$$

$$a \cos \theta = R \cos \theta \cos \alpha$$

$$a = R \cos \alpha \quad (1)$$

$$b \sin \theta = R \sin \theta \sin \alpha$$

$$b = R \sin \alpha \quad (2)$$

$$a^2 + b^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha \quad (\text{Square and add equation 1 and 2})$$

$$a^2 + b^2 = R^2 (\cos^2 \alpha + \sin^2 \alpha)$$

$$\text{The value of } R \text{ is } R = \sqrt{a^2 + b^2}$$

$$\frac{b}{a} = \frac{R \sin \alpha}{R \cos \alpha} \quad (\text{Divide equation 1 and 2})$$

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$$\tan \alpha = \frac{b}{a}$$

$$\text{Therefore, } \alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

**(b) EXPRESSING EQUATION OF THE FORM  $a \cos \theta + b \sin \theta = c$   
INTO THE FORM  $a \cos \theta + b \sin \theta = R \sin(\theta + \alpha)$**

$$a \cos \theta + b \sin \theta = R \sin(\theta + \alpha)$$

$$a \cos \theta + b \sin \theta = R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$$

Compare

$$a \cos \theta = R \cos \theta \sin \alpha$$

$$a = R \sin \alpha$$

(1)

$$b \sin \theta = R \sin \theta \cos \alpha$$

$$b = R \cos \alpha$$

(2)

$$a^2 + b^2 = R^2 \sin^2 \alpha + R^2 \cos^2 \alpha \quad (\text{Square and add equation 1 and 2})$$

$$a^2 + b^2 = R^2 (\sin^2 \alpha + \cos^2 \alpha)$$

$$a^2 + b^2 = R^2 \Rightarrow R = \sqrt{a^2 + b^2}$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{a}{b} \quad (\text{Divide equation 1 by 2})$$

$$\tan \alpha = \frac{a}{b}$$

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$$\alpha = \tan^{-1}\left(\frac{a}{b}\right)$$

$$a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \cos\left(\theta + \tan^{-1}\left(\frac{a}{b}\right)\right)$$

### Questions

1. Express  $a \cos \theta - b \sin \theta = c$  in the form  $R \sin(\theta - \alpha)$
2. Express  $a \cos \theta - b \sin \theta = c$  in the form  $R \cos(\theta + \alpha)$

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### Example 38

Express  $3 \cos \theta + 4 \sin \theta$  in the form  $R \cos(\theta - \alpha)$

### Solution

Compare and expand  $R \cos(\theta - \alpha)$

$$3 \cos \theta + 4 \sin \theta = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$$

$$3 \cos \theta = R \cos \theta \cos \alpha$$

$$3 = R \cos \alpha$$

(1)

$$4 \sin \theta = R \sin \theta \sin \alpha$$

$$4 = R \sin \alpha$$

(2)

$$3^2 + 4^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha$$

$$9+16=R^2(\cos^2 \alpha + \sin^2 \alpha)$$

$$25=R^2 \Rightarrow R=\pm 5$$

$$\frac{4}{3} = \frac{R \sin \alpha}{R \cos \alpha} \Rightarrow \tan \alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$$

$$\text{Therefore, } 3\cos\theta + 4\sin\theta = \pm 5\cos(\theta - 53.1^\circ)$$

#### Example 40

Express  $5\sin\theta - 12\cos\theta = 39$  in the form  $R\sin(\theta - \alpha)$

#### Solution

Compare and expand  $5\sin\theta - 12\cos\theta = R\sin(\theta - \alpha)$

$$5\sin\theta - 12\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

$$5\sin\theta = R\sin\theta\cos\alpha$$

$$5 = R\cos\alpha$$

(1)

$$12\cos\theta = R\cos\theta\sin\alpha$$

$$12 = R\sin\alpha$$

(2)

$$5^2 + 12^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha$$

$$25+144=R^2(\cos^2 \alpha + \sin^2 \alpha)$$

$$169=R^2 \Rightarrow R=\pm 13$$

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$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{12}{5} \Rightarrow \tan \alpha = \frac{12}{5} \Rightarrow \alpha = \tan^{-1}(2.4)$$

$$\alpha = 67.4^\circ$$

$$5 \sin \theta - 12 \cos \theta = \pm 13 \sin(\theta - 67.4^\circ) \text{ but } 5 \sin \theta - 12 \cos \theta = 39$$

$$\pm 13 \sin(\theta - 67.4^\circ) = 39$$

$$\sin(\theta - 67.4^\circ) = \pm 3$$

### Example 41

Write  $\sqrt{2} \cos \theta - \sin \theta = 3$  in the form  $\cos(\theta + \alpha)$  determine the maximum and minimum value.

### Solution

Compare and expand

$$\sqrt{2} \cos \theta - \sin \theta = R \cos(\theta + \alpha)$$

$$\sqrt{2} \cos \theta - \sin \theta = R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$$

$$\sqrt{2} \cos \theta = R \cos \theta \cos \alpha$$

$$\sqrt{2} = R \cos \alpha$$

$$\sin \theta = R \sin \theta \sin \alpha$$

$$1 = R \sin \alpha$$

$$(\sqrt{2})^2 + 1^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha$$

$$2 + 1 = R^2 (\cos^2 \alpha + \sin^2 \alpha)$$

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$$R^2 = 3 \Rightarrow R = \pm\sqrt{3}$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.3^\circ$$

$$\sqrt{2} \cos \theta - \sin \theta = \pm\sqrt{3} \cos(\theta + 35.3^\circ) = 3$$

$$\pm\sqrt{3} \cos(\theta + 35.3^\circ) = 3$$

Therefore,  $\cos(\theta + 35.3^\circ) = \pm \frac{3}{\sqrt{3}}$ , Maximum is  $R = \frac{3}{\sqrt{3}}$  and minimum

$$\text{is } R = -\frac{3}{\sqrt{3}}$$

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### QUESTIONS

(a) Express each of the following in the form  $R \sin(\theta \pm \alpha)$

1.  $\cos \theta + \sqrt{3} \sin \theta = 2$
2.  $3 \sin \theta - 4 \cos \theta = 5$
3.  $4 \cos \theta - 5 \sin \theta = 1$
4.  $10 \cos \theta - 9 \sin \theta = 3$

(b) Express each of the following in the form  $R \cos(\theta \pm \alpha)$

5.  $2 \cos \theta - \sin \theta$
6.  $7 \cos \theta - 10 \sin \theta = 5$
7.  $4 \cos \theta + \sin \theta = 9$
8.  $\sqrt{3} \sin \theta + 2 \cos \theta = 4$

### USING R-FORMULA TO SOLVE TRIGONOMETRIC EQUATION OF THE FORM $a \cos \theta + b \sin \theta = c$

We now clearly know how to express the function  $a \cos \theta \pm b \sin \theta = c$  in the form of  $R \cos(\theta \pm \alpha)$  or  $R \sin(\theta \pm \alpha)$ . This concept is useful in solving trigonometric equation of the form  $a \cos \theta \pm b \sin \theta = c$ .

#### Example 41

Solve  $3 \cos \theta + 4 \sin \theta = 2$  for  $\theta$  from  $0^\circ \leq \theta \leq 360^\circ$

#### Solution

$$3 \cos \theta + 4 \sin \theta = R \cos(\theta - \alpha)$$

$$3 \cos \theta + 4 \sin \theta = R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$$

$$3 \cos \theta = R \cos \theta \cos \alpha$$

$$3 = R \cos \alpha$$

$$4 \sin \theta = R \sin \theta \sin \alpha$$

$$4 = R \sin \alpha$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ$$

$$3^2 + 4^2 = R^2 \cos^2 \alpha + R^2 \sin^2 \alpha$$

$$25 = R^2 \Rightarrow R = 5$$

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$$3\cos\theta + 4\sin\theta = 5\cos(\theta - 53.1^\circ) = 2$$

$$\cos(\theta - 53.1^\circ) = \frac{2}{5}$$

$$\theta - 53.1^\circ = \cos^{-1}(0.4)$$

$$\theta - 53.1^\circ = [66.4^\circ, 293.6^\circ]$$

$$\text{Solution } \theta = 119.5^\circ, 346.7^\circ$$

### Example 42

Find the maximum and minimum value of  $2\sin\theta - 4\cos\theta$  and the corresponding values of  $\theta$  between  $0^\circ$  to  $360^\circ$  inclusive.

#### Solution

$$2\sin\theta - 4\cos\theta = R\sin(\theta - \alpha)$$

$$2\sin\theta - 4\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

$$2\sin\theta = R\sin\theta\cos\alpha$$

$$2 = R\cos\alpha$$

$$4\cos\theta = R\cos\theta\sin\alpha$$

$$4 = R\sin\alpha$$

$$2^2 + 4^2 = R^2(\cos^2\alpha + \sin^2\alpha)$$

$$R = \sqrt{20} = 2\sqrt{5}$$

The minimum value is  $-2\sqrt{5}$  and the maximum value is  $2\sqrt{5}$

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$$\frac{4}{2} = \frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha$$

$$\alpha = \tan^{-1}(2) \Rightarrow \alpha = 63.4^\circ$$

$$2\sqrt{5} \sin(\theta - 63.4^\circ)$$

$$\text{At maximum, } 2\sqrt{5} \sin(\theta - 63.4^\circ) = 2\sqrt{5}$$

$$\theta - 63.4^\circ = \sin^{-1}(1) \Rightarrow \theta - 63.4^\circ = 90^\circ \Rightarrow \theta = 90^\circ + 63.4^\circ = 153.4^\circ$$

$$\text{At minimum, } 2\sqrt{5} \sin(\theta - 63.4^\circ) = -2\sqrt{5}$$

$$\theta - 63.4^\circ = \sin^{-1}(-1) \Rightarrow \theta - 63.4^\circ = 270^\circ \Rightarrow \theta = 333.4^\circ$$

Therefore the solutions are  $(2\sqrt{5}, 153.4^\circ)$  and  $(-2\sqrt{5}, 333.4^\circ)$

### Example 43

Solve  $\cos \theta + \sqrt{5} \sin \theta = 1$  from  $0^\circ \leq \theta \leq 360^\circ$

#### Solution

$$\cos \theta + \sqrt{5} \sin \theta = R \cos(\theta - \alpha)$$

$$\cos \theta + \sqrt{5} \sin \theta = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$$

$$\cos \theta = R \cos \theta \cos \alpha \Rightarrow R \cos \alpha = 1$$

$$\sqrt{5} \sin \theta = R \sin \theta \sin \alpha \Rightarrow \sqrt{5} = R \sin \alpha$$

$$1^2 + (\sqrt{5})^2 = R^2 (\cos^2 \alpha + \sin^2 \alpha)$$

$$6 = R^2 \Rightarrow R = \sqrt{6}$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \sqrt{5} \Rightarrow \tan \alpha = \sqrt{5}$$

$$\alpha = \tan^{-1}(\sqrt{5}) = 65.9^\circ$$

$$R \cos(\theta - \alpha) = \sqrt{6} \cos(\theta - 65.9^\circ)$$

$$\sqrt{6} \cos(\theta - 65.9^\circ) = 1$$

$$\theta - 65.9^\circ = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$$

$$\theta - 65.9^\circ = 65.9^\circ$$

The solution,  $\theta = 0^\circ, 360^\circ$

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### EXERCISE 8

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(a) Solve each of the following using R-formula from  $0^\circ$  to  $360^\circ$  inclusive

1.  $3\cos\theta + 2\sin\theta = 5$
2.  $\sqrt{2}\sin\theta + \sqrt{2}\cos\theta = 3$
3.  $\cos 3\theta + \sin 3\theta = 1$
4.  $5\sin 2\theta - 12\cos 2\theta = 7$
5.  $\cos\left(\frac{1}{2}x\right) - 3\sin\left(\frac{1}{2}x\right) = 2$
6.  $2\cos\theta + 3\sin\theta = 7$
7.  $\frac{1}{4}\cos\theta + \frac{1}{3}\sin\theta = 1$
8.  $\cos 2\theta + \sqrt{3}\sin 2\theta + 2 = 0$
9.  $(2\cos\theta - 3\sin\theta)^2 = 3$
10.  $\sqrt{3}\cos(2\theta) + \sqrt{5}\sin 2\theta = 10$
11.  $\sin\theta\cos\theta = 4$
12.  $\sin 3\theta\cos 3\theta = 0$

13.  $5\sin\theta - 12\cos\theta = 6$

14.  $\sin\theta + \cos\theta = \frac{1}{2}$

15.  $\cos\theta - 7\sin\theta = 2$

16.  $2\sin\theta + 7\cos\theta = 4$

17.  $3\tan\theta - 2\sec\theta = 4$

18.  $4\sin\theta\cos\theta + 15\cos 2\theta = 10$

19.  $\cos\theta + 2\sin\theta = 0$

20.  $\sqrt{3}\sin\theta + \cos\theta = 1$

(b) Find the maximum and minimum value of the expressions below and find the corresponding value of  $\theta$

21.  $\cos\theta + \sin\theta$

22.  $2\cos\theta + 3\sin\theta$

23.  $\sqrt{2}\cos\theta - \sin\theta$

24.  $\cos\theta - \sqrt{5}\sin\theta$

25.  $2\cos\theta - \sqrt{3}\sin\theta$

26.  $3\cos\theta - \sqrt{2}\sin\theta$

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## GENERAL SOLUTION TO THE TRIGONOMETRIC EQUATIONS

The solution of trigonometric equations are divided into two major parts

1. Principal solution, are the solution which take an angle form  $0^\circ$  to  $360^\circ$  inclusively.
2. General solution, these are all possible solution of the trigonometric equations
  - (a) If  $\sin \theta = \sin \alpha$ , then  $\theta = \pi n + (-1)^n \alpha$
  - (b) If  $\cos \theta = \cos \alpha$ , then  $\theta = 2\pi n \pm \alpha$
  - (c) If  $\tan \theta = \tan \alpha$ , then  $\theta = \pi n + \alpha$

For all three cases above,  $n \in \mathbb{Z}$  ( $n$  is any integer)

### Example 44

Find the general solution of  $\sin \alpha = -1$

#### Solution

Given  $\sin \alpha = -1$

$$\sin \alpha = -1 \Rightarrow \theta = \sin^{-1}(-1)$$

$$\alpha = -90^\circ = -\frac{\pi}{2}$$

$$\theta = \pi n + (-1)^n \alpha$$

The general solution is  $\theta = \pi n - (-1)^n \left( \frac{\pi}{2} \right)$

### Example 45

Find the general solution of  $\tan x = \sqrt{3}$

#### Solution

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$$\tan x = \sqrt{3}$$

$$x = \tan^{-1}(\sqrt{3})$$

$$x = \frac{\pi}{3}$$

The general solution is  $\theta = 2\pi n + \frac{\pi}{3}$ , for  $n \in \mathbb{Z}$

#### Example 46

Find the general solution  $\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

#### Solution

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$x + \frac{\pi}{4} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$x + \frac{\pi}{4} = \frac{\pi}{4}$$

$$x = 0$$

The general solution is  $\theta = \pi n$

#### Example 47

Find the general solution of  $\cos x + 3\sin x = 2$

#### Solution

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$$\cos x + 3\sin x = 2$$

$$\cos x + 3\sin x = R\cos(x - \alpha)$$

$$\cos x + 3\sin x = R\cos x \cos \alpha + R\sin x \sin \alpha$$

$$\cos x = R\cos x \cos \alpha$$

$$R\cos \alpha = 1$$

$$3\sin x = R\sin x \sin \alpha$$

$$R\sin \alpha = 3$$

$$R = \sqrt{1^2 + 3^2} \Rightarrow R = \sqrt{10}$$

$$\tan \alpha = 3 \Rightarrow \alpha = \tan^{-1}(3)$$

$$\alpha = 71.56^\circ$$

$$\cos x + 3\sin x = \sqrt{10} \cos(x - 71.56^\circ)$$

$$\sqrt{10} \cos(x - 71.56^\circ) = 2$$

$$\cos(x - 71.56^\circ) = \frac{2}{\sqrt{10}}$$

$$x - 71.56^\circ = \cos^{-1}\left(\frac{2}{\sqrt{10}}\right)$$

The general solution is  $x - 71.56^\circ = 2\pi n \pm 50.77^\circ$

$$x = 2\pi n + 122.33^\circ \text{ or } x = 2\pi n - 20.79^\circ$$

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**EXERCISE 9**

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Find the general solution of the following trigonometric equations

1.  $\sin x = -\frac{1}{2}$
2.  $\cos 2x = \frac{1}{\sqrt{2}}$
3.  $\tan x = -1$
4.  $\sin\left(x - \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}}$
5.  $\tan\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right)$
6.  $\tan x = \sqrt{3}$
7.  $\sin^2 x = \frac{1}{2}$
8.  $\cos\left(x - \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}$
9.  $\cos\left(x + \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$
10.  $\cos x + 2\sin x = 2$

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**FACTOR FORMULA**

Consider the sine compound angles

$$+ \begin{cases} \sin(A+B) = \sin A \cos B + \cos A \sin B \\ \sin(A-B) = \sin A \cos B - \cos A \sin B \end{cases}$$

$$\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$$

Let  $A + B = C$  and  $A - B = D$

$$\pm \begin{cases} A + B = C \\ A - B = D \end{cases}$$

$$\text{Then } A = \frac{C + D}{2} \text{ and } B = \frac{C - D}{2}$$

$$\text{The equation become, } \sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$\text{Also, from } - \begin{cases} \sin(A + B) = \sin A \cos B + \cos A \sin B \\ \sin(A - B) = \sin A \cos B - \cos A \sin B \end{cases}$$

$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$$

Substituting the values of  $A + B$  and  $A - B$ ,

$$\sin C - \sin D = 2 \cos\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$$

Consider the cosine compound angles

$$+ \begin{cases} \cos(A + B) = \cos A \cos B - \sin A \sin B \\ \cos(A - B) = \cos A \cos B + \sin A \sin B \end{cases}$$

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$

$$\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$\text{Also, from } - \begin{cases} \cos(A + B) = \cos A \cos B - \sin A \sin B \\ \cos(A - B) = \cos A \cos B + \sin A \sin B \end{cases}$$

$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$$

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$$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

### FACTOR FORMULA SUMMARY

#### Sine

$$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right) \text{ and}$$

$$\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

#### Cosine

$$\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right) \text{ and}$$

$$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

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### PROVING IDENTITIES INVOLVING FACTOR FORMULA

#### Example 48

Prove that  $\frac{\sin B - \sin C}{\sin B + \sin C} = \cot\left(\frac{B+C}{2}\right) \tan\left(\frac{B-C}{2}\right)$

#### Solution

$$\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2 \cos\left(\frac{B+C}{2}\right) \sin\left(\frac{B-C}{2}\right)}{2 \sin\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right)}$$

$$\begin{aligned}
&= \left( \frac{\cos\left(\frac{B+C}{2}\right)}{\sin\left(\frac{B+C}{2}\right)} \right) \left( \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)} \right) \\
&= \cot\left(\frac{B+C}{2}\right) \tan\left(\frac{B-C}{2}\right) \text{ hence shown}
\end{aligned}$$

**Example 49**

Prove that  $\sin A + \sin 2A + \sin 3A = \sin 2A(2\cos A + 1)$

**Solution**

$$\sin 2A + (\sin A + \sin 3A)$$

$$\sin 2A + 2\sin\left(\frac{A+3A}{2}\right)\cos\left(\frac{A-3A}{2}\right)$$

$$\sin 2A + 2\sin 2A \cos A$$

$$\sin 2A(1 + 2\cos A)$$

Remember that

$$\cos(-A) = \cos(A)$$

**Example 50**

Prove that  $\sin 2A - \sin 4A = -2\cos 3A \sin A$

**Solution**

$$\sin 2A - \sin 4A = -2\cos 3A \sin A$$

$$\sin 2A - \sin 4A = 2\cos\left(\frac{2A+4A}{2}\right)\sin\left(\frac{2A-4A}{2}\right)$$

$$= 2\cos(3A)\sin(-A)$$

$$= -2\cos 3A \sin A$$

**Example 51**

Show that  $\frac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \tan\left(\frac{A}{2}\right)$  (not concerned with factor formula)

**Solution**

$$\begin{aligned}
 \frac{1 + \sin A - \cos A}{1 + \sin A + \cos A} &= \frac{(1 - \cos A) + \sin A}{(1 + \cos A) + \sin A} \\
 &= \frac{\left[1 - \cos^2\left(\frac{A}{2}\right) + \sin^2\left(\frac{A}{2}\right)\right] + \left[2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)\right]}{\left[1 + \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)\right] + \left[2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)\right]} \\
 &= \frac{\left[2 \sin^2\left(\frac{A}{2}\right)\right] + \left[2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)\right]}{\left[2 \cos^2\left(\frac{A}{2}\right)\right] + \left[2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)\right]} \\
 &= \frac{2 \sin\left(\frac{A}{2}\right) \left[\sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)\right]}{2 \cos\left(\frac{A}{2}\right) \left[\sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)\right]} \\
 &= \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)} = \tan\left(\frac{A}{2}\right) \text{ Hence shown}
 \end{aligned}$$

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**Example 52**

Prove that  $\sin(150^\circ + A) + \sin(150^\circ - A) = \cos A$

**Solution**

Left hand side

$$\sin(150^\circ + A) + \sin(150^\circ - A) = \cos A$$

$$2 \sin\left(\frac{(150^\circ + A) + (150^\circ - A)}{2}\right) \cos\left(\frac{(150^\circ + A) - (150^\circ - A)}{2}\right)$$

$$= 2 \sin\left(\frac{300^\circ}{2}\right) \cos\left(\frac{2A}{2}\right)$$

$$= 2 \sin(150^\circ) \cos A$$

$$= 2 \left(\frac{1}{2}\right) \cos A = \cos A \text{ Hence shown.}$$

**Example 53**

Prove that  $\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \tan A$

**Solution**

Consider left hand side

$$\frac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = \frac{2 \cos\left(\frac{5A + 3A}{2}\right) \sin\left(\frac{5A - 3A}{2}\right)}{2 \cos\left(\frac{3A + 5A}{2}\right) \cos\left(\frac{3A - 5A}{2}\right)}$$

$$= \frac{\cos(4A) \sin(A)}{\cos(4A) \cos(A)} = \frac{\sin A}{\cos A} = \tan A$$

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**Example 54**

If  $A + B + C = \pi$ , show that

$$\cos A + \cos B + \cos C = 1 + 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)$$

**Solution**

Consider left hand side

$$\cos A + \cos B + \cos C = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) + \left(\cos^2\left(\frac{C}{2}\right) - \sin^2\left(\frac{C}{2}\right)\right)$$

$$\Rightarrow 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) + \left(1 - 2 \sin^2\left(\frac{C}{2}\right)\right)$$

$$\text{But } \frac{A+B}{2} = \frac{\pi - C}{2}$$

$$\Rightarrow 2 \cos\left(\frac{\pi - C}{2}\right) \cos\left(\frac{A-B}{2}\right) + 1 - 2 \sin^2\left(\frac{C}{2}\right)$$

$$\Rightarrow 1 + 2 \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) \cos\left(\frac{A-B}{2}\right) - 2 \sin^2\left(\frac{C}{2}\right)$$

$$\Rightarrow 1 + 2 \sin\left(\frac{C}{2}\right) \cos\left(\frac{A-B}{2}\right) - 2 \sin^2\left(\frac{C}{2}\right)$$

$$\Rightarrow 1 + 2 \sin\left(\frac{C}{2}\right) \left[ \cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right) \right]$$

$$\text{But } \frac{A+B}{2} + \frac{C}{2} = \frac{\pi}{2} \Rightarrow \frac{C}{2} = \frac{\pi}{2} - \left(\frac{A+B}{2}\right)$$

$$\Rightarrow 1 + 2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{\pi}{2} - \left(\frac{A+B}{2}\right)\right)\right]$$

$$\Rightarrow 1 + 2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$\Rightarrow 1 + 2\sin\left(\frac{C}{2}\right)\left[-2\sin\left(\frac{A}{2}\right)\sin\left(-\frac{B}{2}\right)\right]$$

$$\Rightarrow 1 + 2\sin\left(\frac{C}{2}\right)\left[2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right]$$

$$\Rightarrow 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

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## SIMPLIFYING TRIGONOMETRIC IDENTITIES INVOLVING FACTOR FORMULA

### Example 55

Simplify  $\frac{\sin 8A \cos A - \sin 6A \cos 3A}{\cos 2A \cos A - \sin 3A \sin 4A}$

### Solution

$$\begin{aligned} &= \frac{\frac{1}{2}[\sin(8A + A) + \sin(8A - A)] - \frac{1}{2}[\sin(6A + 3A) + \sin(6A - 3A)]}{\frac{1}{2}[\cos(2A + A) + \cos(2A - A)] + \frac{1}{2}[\cos(3A + 4A) - \cos(3A - 4A)]} \\ &= \frac{\sin 9A + \sin 7A - \sin 9A - \sin 3A}{\cos 3A + \cos A + \cos 7A - \cos A} \\ &= \frac{\sin 7A - \sin 3A}{\cos 7A + \cos 3A} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2 \cos\left(\frac{7A+3A}{2}\right) \sin\left(\frac{7A-3A}{2}\right)}{2 \cos\left(\frac{7A+3A}{2}\right) \cos\left(\frac{7A-3A}{2}\right)} \\
 &= \frac{\sin 2A}{\cos 2A} = \tan 2A
 \end{aligned}$$

### Example 56

Simplify  $(\cos A + \cos B)^2 + (\sin A + \sin B)^2$

#### Solution

$$\begin{aligned}
 &(\cos A + \cos B)^2 + (\sin A + \sin B)^2 \\
 &(\cos A + \cos B)^2 + (\sin A + \sin B)^2 \\
 &= \left(2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)\right)^2 + \left(2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)\right)^2 \\
 &= 4 \cos^2\left(\frac{A+B}{2}\right) \cos^2\left(\frac{A-B}{2}\right) + 4 \sin^2\left(\frac{A+B}{2}\right) \cos^2\left(\frac{A-B}{2}\right) \\
 &= 4 \cos^2\left(\frac{A-B}{2}\right) \left[ \cos^2\left(\frac{A+B}{2}\right) + \sin^2\left(\frac{A+B}{2}\right) \right] \\
 &= 4 \cos^2\left(\frac{A-B}{2}\right)
 \end{aligned}$$

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## EXERCISE 10

1. Prove that  $\frac{\cos(4A) - \cos(2A)}{\sin(4A) + \sin(2A)} = -\tan A$
2. Show that  $\frac{\sin 4A + \sin 2A}{\cos 4A + \cos 2A} = \tan 3A$
3. Show that  $\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}$
4. Show that  $\sin 50^\circ + \sin 40^\circ = \sqrt{2} \cos 5^\circ$
5. Prove that  $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{1}{2}(A + B)$
6. Show that  $\cos^3 A \sin^2 A = \frac{1}{16} (2 \cos A - \cos 3A - \cos 5A)$
7. Prove that  $1 + \cos(2A) + \cos(4A) + \cos(6A) = 4 \cos(A) \cos(2A) \cos(3A)$
8. Show that  $\sin \theta + \sin 2\theta + \sin 3\theta = \sin 2\theta (1 + 2 \cos \theta)$
9. Simplify  $\frac{\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta}{\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta}$
10. Simplify (a)  $\frac{\sin 4A + \sin 2A}{\cos A}$  (b)  $\frac{\cos\left(\frac{3}{2}A\right) - \cos\left(\frac{9}{2}A\right)}{2 \sin 3A}$
11. Simplify  $\frac{\sin 7A - \sin 3A}{\cos 5A \sin 2A}$
12. Prove that  $\frac{\cos 50^\circ - \cos 70^\circ}{\sin 70^\circ - \sin 50^\circ} = \sqrt{3}$
13. Simplify  $\frac{\cos 50^\circ + \cos 70^\circ}{\sin 70^\circ + \sin 50^\circ}$

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## SOLVING TRIGONOMETRIC INVOLVING FACTOR FORMULA

**Example 57**

Find the general solution of  $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

**Solution**

$$\sin\theta + \sin 3\theta + \sin 5\theta = 0$$

$$(\sin\theta + \sin 5\theta) + \sin 3\theta = 0$$

$$2\sin\left(\frac{\theta + 5\theta}{2}\right)\cos\left(\frac{\theta - 5\theta}{2}\right) + \sin 3\theta = 0$$

$$2\sin 3\theta \cos 2\theta + \sin 3\theta = 0$$

$$\sin 3\theta (2\cos 2\theta + 1) = 0$$

$$\sin 3\theta = 0$$

$$3\theta = \sin^{-1}(0)$$

$$3\theta = \pi n + (-1)^n \alpha$$

$$3\theta = \pi n \Rightarrow \theta = \frac{\pi n}{3}$$

$$\cos 2\theta = -\frac{1}{2}$$

$$2\theta = \cos^{-1}(-0.5)$$

$$2\theta = 120^\circ = \frac{2\pi}{3}$$

$$2\theta = 2\pi m \pm \alpha$$

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$$\theta = m\pi \pm \frac{\pi}{3}$$

The general solution set is  $\theta = m\pi \pm \frac{\pi}{3}$  or  $\theta = \frac{n\pi}{3}$

### Example 58

Find the general solution of  $\sin\theta + \sin 5\theta = 0$

#### Solution

Given  $\sin\theta + \sin 5\theta = 0$

$$\sin\theta + \sin 5\theta = 2\sin\left(\frac{\theta + 5\theta}{2}\right)\cos\left(\frac{\theta - 5\theta}{2}\right) = 0$$

$$= 2\sin 3\theta \cos 2\theta = 0$$

Case 1:  $\sin 3\theta = 0$

$$= 3\theta = \sin^{-1}(0)$$

$$3\theta = n\pi$$

The general solution is  $\theta = \frac{n\pi}{3}$

Case 2:  $\cos 2\theta = 0$

$$2\theta = \cos^{-1}(0)$$

$$2\theta = \frac{\pi}{2}$$

$$2\theta = 2n\pi \pm \frac{\pi}{2}$$

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The general solution set is  $\theta = n\pi \pm \frac{\pi}{4}$

### Example 59

Find the general solution of  $\cos(x + 20^\circ) - \cos(x - 40^\circ) = 0$

#### Solution

$$\cos(x + 20^\circ) - \cos(x - 40^\circ) = 0$$

$$\cos(x + 20^\circ) - \cos(x - 40^\circ) = -2 \sin\left(\frac{2x - 20^\circ}{2}\right) \sin\left(\frac{60^\circ}{2}\right) = 0$$

$$= -2 \left(\frac{1}{2}\right) \sin(x - 10^\circ) = 0$$

$$\sin(x - 10^\circ) = 0$$

$$x - 10^\circ = \sin^{-1}(0)$$

$$x - 10^\circ = n\pi$$

The general solution is  $x = n\pi + \frac{\pi}{18}$

### Example 60

Find the general solution of  $\cos(2\theta + 70^\circ) + \cos(2\theta - 20^\circ) = 0$

#### Solution

$$\cos(2\theta + 70^\circ) + \cos(2\theta - 20^\circ) = 0$$

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$$\cos(2\theta + 20^\circ) + \cos(2\theta - 40^\circ) = -2 \sin\left(\frac{4\theta - 50^\circ}{2}\right) \sin\left(\frac{90^\circ}{2}\right)$$

$$-2 \sin 45^\circ \sin(2\theta - 25^\circ) = 0$$

$$\sin(2\theta - 25^\circ) = 0$$

$$2\theta - 25^\circ = \sin^{-1}(0)$$

$$2\theta - 25^\circ = n\pi$$

$$2\theta = n\pi + 25^\circ$$

$$\text{The general solution is } \theta = \frac{n\pi}{2} + \frac{5\pi}{72}$$

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### EXERCISE 11

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- Given that  $\sin 2A + \sin 2B = x$  and  $\cos 2A + \cos 2B = y$ , show that  $\frac{x}{y} = \tan(A + B)$ . Prove that  $\frac{4x}{x^2 + y^2 + 2y} = \frac{\sin(A + B)}{\cos A \cos B}$  and deduce an expression for  $\tan A \tan B$  in terms of  $x$  and  $y$
- Show that if  $\sin \theta = \frac{1-y}{1+y}$  then  $\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sqrt{y}$
- If  $\tan 3A = 2$  simplify  $\frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A}$
- If  $\sin A + \sin B = p$  and  $\cos A + \cos B = q$ , show that
  - $\sin(A + B) = \frac{2pq}{p^2 + q^2}$

$$(b) \cos(A+B) = \frac{q^2 - p^2}{q^2 + p^2}$$

$$(c) \tan A + \tan B = \frac{8pq}{q^2 - 3p^2}$$

$$(d) \cos 2A + \cos 2B = \frac{(q^2 - p^2)(p^2 + q^2 - 2)}{p^2 + q^2}$$

5. Find the general solution of each of the following

$$(a) \cos 3\theta + \cos \theta = 0$$

$$(b) \cos 2\theta + \cos 4\theta = \cos 6\theta$$

$$(c) \sin(3\theta + 10^\circ) + \sin(3\theta - 10^\circ) = 0$$

$$(d) \cos \theta - \cos 5\theta = 0$$

$$(e) \sin \theta + \sin 3\theta + \sin 5\theta = 0$$

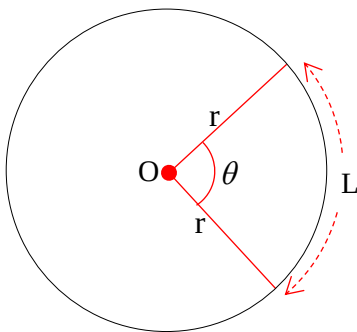
$$(f) \cos \frac{1}{2}\theta + 2\cos \frac{3}{2}\theta + \cos \frac{5}{2}\theta = 0$$

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## RADIANS AND SMALL ANGLES

An angle can be measured in degree or radians.

Consider the diagrams below



Using ration theorem

$$\frac{l}{c} = \frac{\theta}{360^\circ}$$

$$\text{But } c = 2\pi r$$

$$\frac{l}{2\pi r} = \frac{\theta}{360^\circ}$$

$$\text{rad} = \frac{\pi r}{180^\circ}$$

$$360^\circ l = 2\pi r \theta$$

$$\frac{l}{r} = \frac{\pi \theta}{180^\circ}$$

$$\frac{l}{r} = \text{rad} = \frac{\pi r}{180^\circ}$$

The radian formula

### Example 61

Express the following in radians

- (a)  $45^\circ$                       (b)  $60^\circ$               (c)  $70^\circ$

**Solution**

$$(a) \text{ rad} = \frac{\pi r}{180^\circ}$$

$$45^\circ = \frac{\pi(45^\circ)}{180^\circ} = \frac{\pi}{4}$$

$$(b) 60^\circ = \frac{\pi(60^\circ)}{180^\circ} = \frac{\pi}{3}$$

$$(c) 70^\circ = \frac{\pi(70^\circ)}{180^\circ} = \frac{7\pi}{18}$$

### Example 62

Express the following in degree

(a)  $\frac{2\pi}{3}$

(b)  $\frac{\pi}{8}$

(c)  $\frac{5\pi}{7}$

**Solution**

$$\text{deg} = \frac{180^\circ \text{rad}}{\pi}$$

$$(a) \text{ deg} = \left( \frac{180^\circ}{\pi} \right) \left( \frac{2\pi}{3} \right) = 120^\circ$$

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$$(b) \deg = \left( \frac{180^\circ}{\pi} \right) \left( \frac{\pi}{8} \right) = 22.5^\circ$$

$$(c) \deg = \left( \frac{180^\circ}{\pi} \right) \left( \frac{5\pi}{7} \right) = 128^\circ 34'$$

### SMALL ANGLES

Let refer the Maclaurin's series for **sine, cosine and tangent**, through these series we deduce the approximations for small angles as  $\theta \rightarrow 0$ .

#### Sine

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$\lim_{\theta \rightarrow 0} (\sin \theta) = \lim_{\theta \rightarrow 0} \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right)$$

Ignoring third and higher derivatives, we get

$$\lim_{\theta \rightarrow 0} (\sin \theta) \approx \theta$$

$$\lim_{\theta \rightarrow 0} (\sin \theta) \approx \theta$$

#### Cosine

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\lim_{\theta \rightarrow 0} (\cos \theta) = \lim_{\theta \rightarrow 0} \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \right)$$

Ignoring the higher powers

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$$\lim_{\theta \rightarrow 0} (\cos \theta) \approx 1 - \frac{\theta^2}{2!}$$

$$\lim_{\theta \rightarrow 0} (\cos \theta) \approx 1 - \frac{\theta^2}{2}$$

### Tangent

$$\tan \theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \frac{17}{315}\theta^7 + \dots$$

$$\lim_{\theta \rightarrow 0} (\tan \theta) = \lim_{\theta \rightarrow 0} \left( \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \frac{17}{315}\theta^7 + \dots \right)$$

Ignoring the higher power

$$\lim_{\theta \rightarrow 0} (\tan \theta) \approx \theta$$

$$\lim_{\theta \rightarrow 0} (\tan \theta) \approx \theta$$

### Example 63

Simplify  $\frac{1 - \cos 4\theta}{\sin \theta \tan \theta}$  if  $\theta$  is small

### Solution

$$\frac{1 - \cos 4\theta}{\sin \theta \tan \theta} = \frac{1 - \left( 1 - \frac{(4\theta)^2}{2} \right)}{\theta^2} = \frac{16\theta^2}{2\theta^2} = 8$$

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**Example 64**

Evaluate the following as  $\theta \rightarrow 0$

- (a)  $\frac{\theta \tan \theta}{1 - \cos 2\theta}$   
 (b)  $\frac{\sin(\alpha + \theta) + \sin \alpha}{\sin \theta}$

**Solution**

$$(a) \lim_{\theta \rightarrow 0} \left( \frac{\theta \tan \theta}{1 - \cos 2\theta} \right) = \frac{\theta^2}{1 - \left( 1 - \frac{(2\theta)^2}{2} \right)} = \frac{\theta^2}{2\theta^2} = \frac{1}{2}$$

$$\lim_{\theta \rightarrow 0} \left( \frac{\theta \tan \theta}{1 - \cos 2\theta} \right) \approx \frac{1}{2}$$

$$(b) \lim_{\theta \rightarrow 0} \left( \frac{\sin(\alpha + \theta) + \sin \alpha}{\sin \theta} \right) \theta = \left( \frac{\sin \theta \cos \alpha + \cos \theta \sin \alpha}{\sin \theta} \right) \theta$$

$$= \left( \frac{\theta \cos \alpha + \left( 1 - \frac{\theta^2}{2} \right) \sin \alpha}{\theta} \right) \theta$$

$$\lim_{\theta \rightarrow 0} \left( \frac{\sin(\alpha + \theta) + \sin \alpha}{\sin \theta} \right) \theta \approx \cos \alpha$$

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## TRIGONOMETRIC FUNCTION

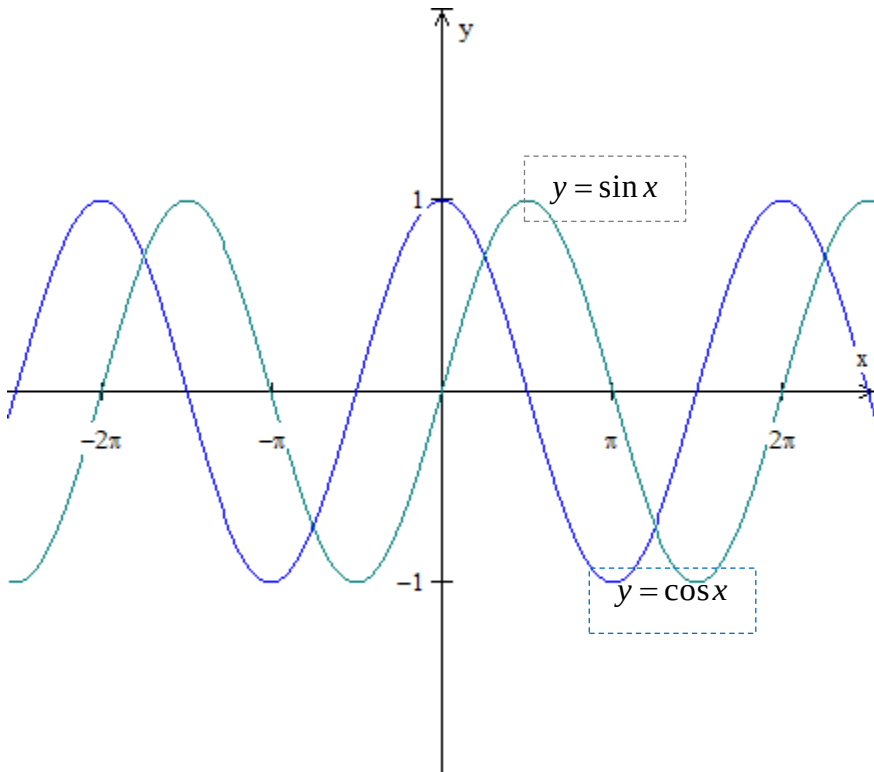
As other functions, in trigonometric functions we can also determine domain and range of a function. Where  $x$  being values which gives out an output of the function.

Consider the function  $f(x) = \cos x$

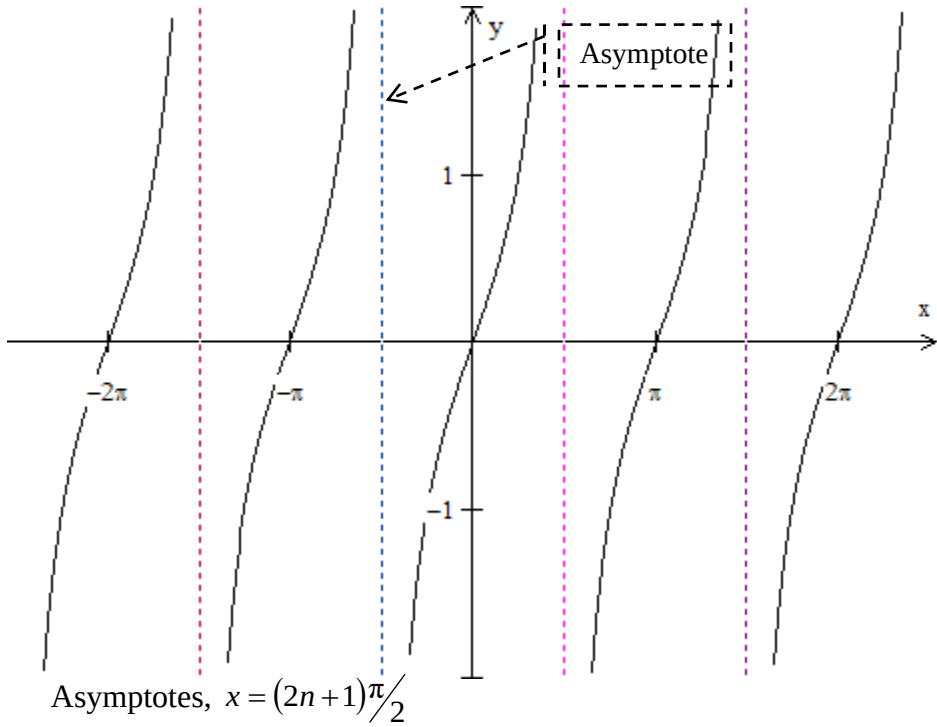
Domain of  $f(x) = \cos x$  is the set of all real numbers ( $x$  is in radian) and range is any number in the intervals,  $-1 \leq y \leq 1$  thus  $|y| \leq 1$ . The domain and range of cosine are the same for sine.

For tangent, domain are all values except  $(2n+1)\frac{\pi}{2}$ , where  $n$  is any integer, range is a set of all real numbers.

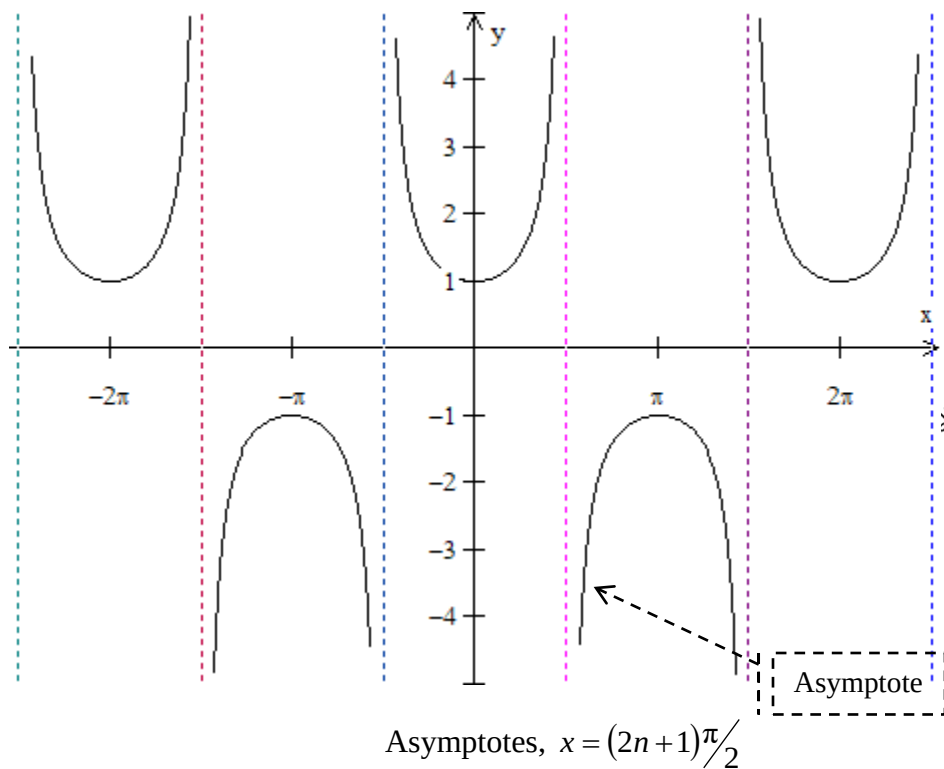
**Graph of sine and cosine**



## Graph of tangent

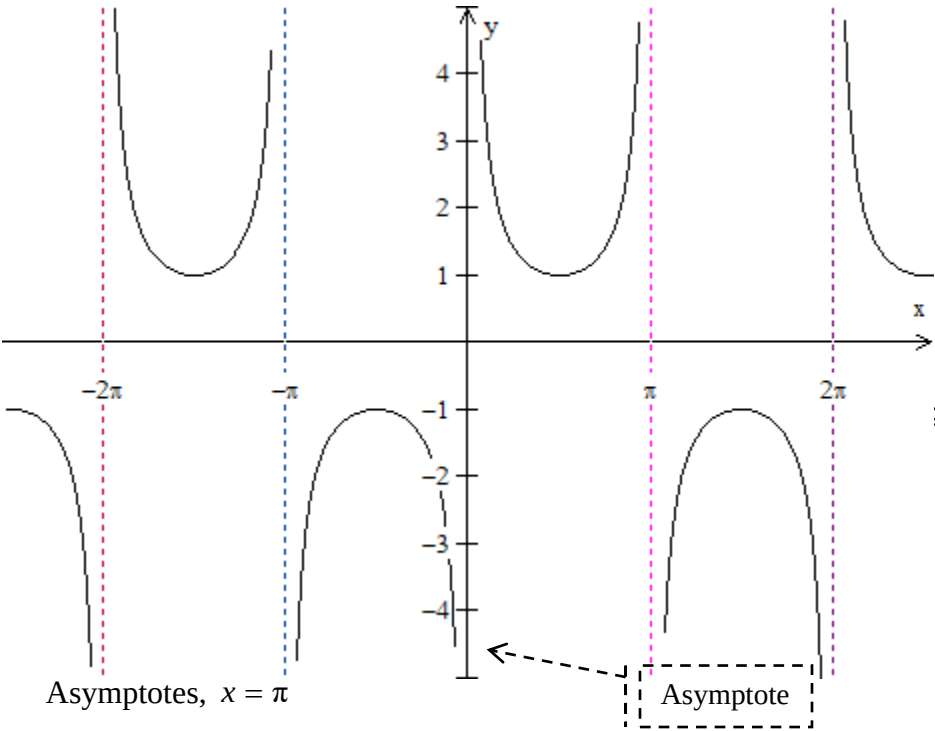


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**Graph of secant**

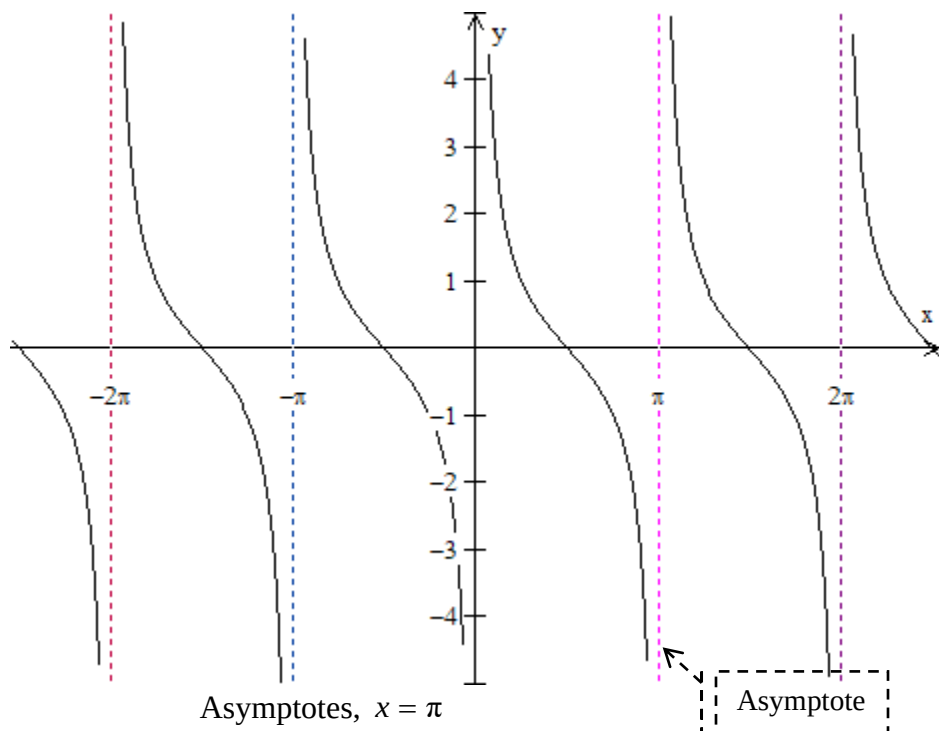
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Graph of cosecant



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## Graph of cotangent

**Example 65**

Determine the domain and range of  $y = 3 + 2 \cos x$  hence sketch the graph.

**Solution**

Using the standard domain and range

Range:  $-1 \leq y \leq 1$  this is the standard range

$$y = 3 + 2 \cos x$$

$$-1 \leq \cos x \leq 1$$

Multiply by 2 through out

$$-2 \leq 2 \cos x \leq 2$$

Add 3 both sides

$$-2 + 3 \leq 3 + 2\cos x \leq 2 + 3$$

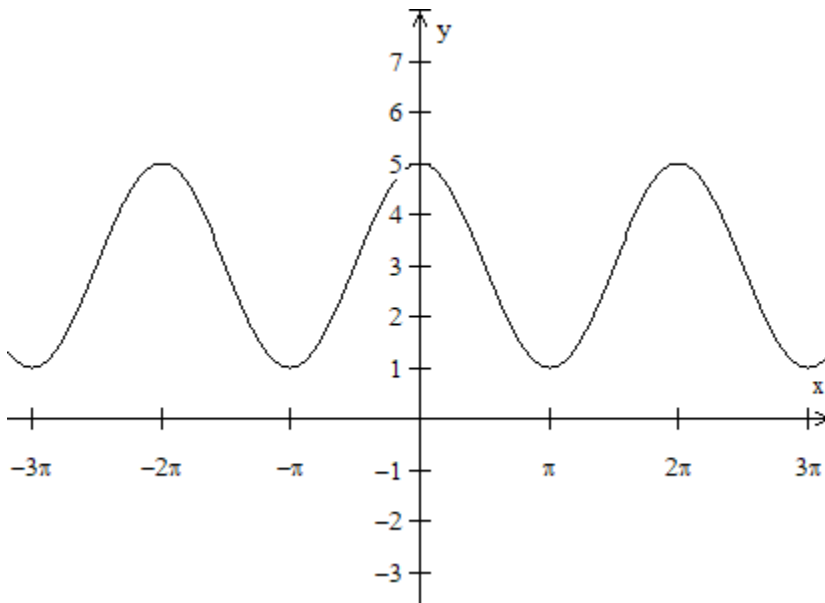
Graph of  $y = 3 + 2\cos x$

The range becomes

$$1 \leq 3 + 2\cos x \leq 5 \Rightarrow 1 \leq y \leq 5$$

The range set is  $1 \leq y \leq 5$

$$\text{Domain} = \{x : x \in \mathbb{R}\}$$



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### Example 66

Determine the domain and range of the function  $f(x) = -4 + \sin 2x$

**Solution**

Given,  $f(x) = -4 + \sin 2x$

$$\text{Domain} = \{x : x \in \mathbb{R}\}$$

Range

Standard range,  $-1 \leq y \leq 1$

$$y = f(x) = -4 + \sin 2x$$

$$-1 \leq \sin 2x \leq 1$$

Subtract 4 both sides

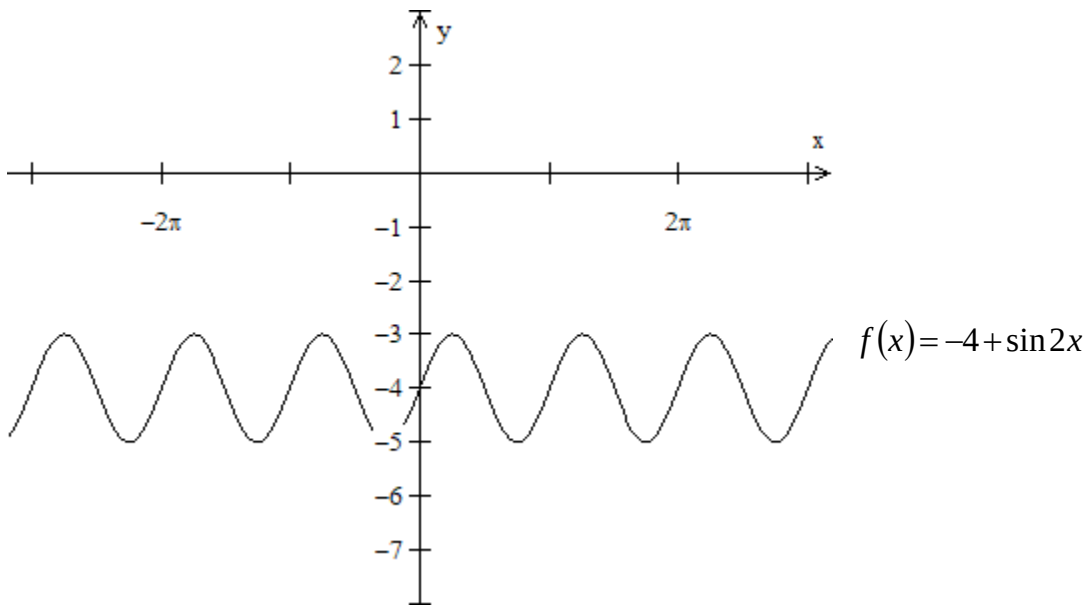
$$-4 - 4 \leq -4 + \sin 2x \leq 1 - 4$$

$$-5 \leq -4 + \sin x \leq -3$$

$$-5 \leq y \leq -3$$

$$\text{Range} = \{y : -5 \leq y \leq -3\}$$

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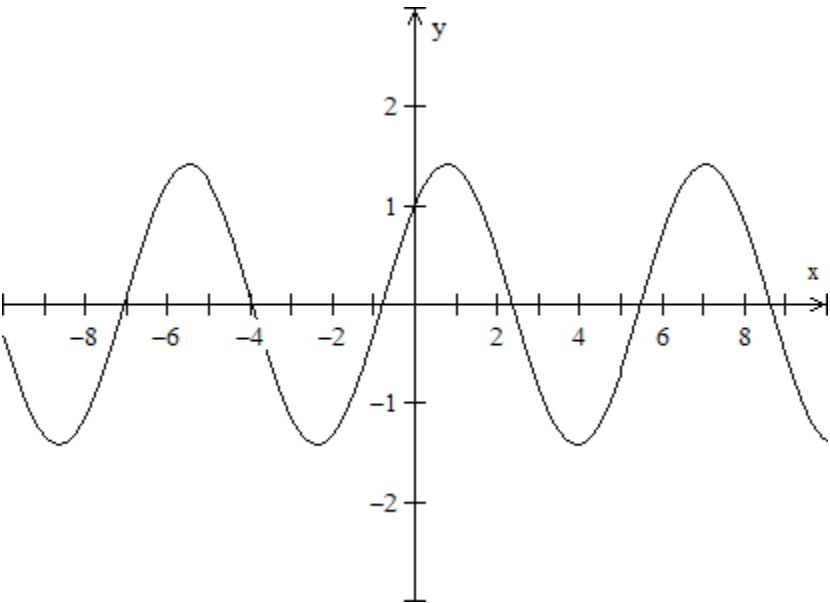
Example 67

Sketch the graph of  $f(x) = \sin x + \cos x$

Solution

Table of values

|   |      |       |       |       |      |      |      |       |       |
|---|------|-------|-------|-------|------|------|------|-------|-------|
| x | -4   | -3    | -2    | -1    | 0    | 1    | 2    | 3     | 4     |
| y | 0.10 | -1.13 | -1.33 | -0.30 | 1.00 | 1.38 | 0.49 | -0.85 | -1.41 |



$f(x) = \sin x + \cos x$

## PROVING EQUATION INVOLVING INVERSE OF TRIGONOMETRIC FUNCTIONS

### Example 68

Prove that  $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) + \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) = \tan^{-1}(\sqrt{3})$

### Solution

$$\text{Given } \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) + \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) = \tan^{-1}(\sqrt{3})$$

$$\text{Let } A = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \Rightarrow \tan A = \frac{\sqrt{3}}{2} \text{ and } B = \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) \Rightarrow \tan B = \frac{\sqrt{3}}{5}$$

$$\text{Therefore, } A + B = \tan^{-1}(\sqrt{3})$$

Introducing tan

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A + B) = \frac{\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{5}\right)}{1 - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{5}\right)}$$

$$\tan(A + B) = \frac{\frac{5\sqrt{3} + 2\sqrt{3}}{10}}{1 - \frac{3}{10}} = \frac{7\sqrt{3}/10}{7/10} = \frac{7\sqrt{3}}{10} \times \frac{10}{7} = \sqrt{3}$$

$$\tan(A + B) = \sqrt{3}$$

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Then  $A + B = \tan^{-1}(\sqrt{3})$

But A and B are  $A = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$  and  $B = \tan^{-1}\left(\frac{\sqrt{3}}{5}\right)$

Hence,  $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) + \tan^{-1}\left(\frac{\sqrt{3}}{5}\right) = \tan^{-1}(\sqrt{3})$

### Example 69

Prove that  $\sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \sin^{-1}\left(\frac{63}{65}\right)$

### Solution

Let  $\sin^{-1}\left(\frac{5}{13}\right) = A \Rightarrow \sin A = \frac{5}{13}$  and  $\sin^{-1}\left(\frac{4}{5}\right) = B \Rightarrow \sin B = \frac{4}{5}$

$$A + B = \sin^{-1}\left(\frac{63}{65}\right)$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A + B) = \sin A \sqrt{1 - \sin^2 B} + \sqrt{1 - \sin^2 A} \sin B$$

$$\sin(A + B) = \left(\frac{5}{13}\right) \left(\sqrt{1 - \left(\frac{4}{5}\right)^2}\right) + \left(\sqrt{1 - \left(\frac{5}{13}\right)^2}\right) \left(\frac{4}{5}\right)$$

$$\sin(A + B) = \left(\frac{5}{13}\right) \left(\frac{3}{5}\right) + \left(\frac{4}{5}\right) \left(\frac{12}{13}\right)$$

$$\sin(A + B) = \frac{15}{65} + \frac{48}{65}$$

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$$\sin(A+B) = \frac{63}{65}$$

$$A+B = \sin^{-1}\left(\frac{63}{65}\right)$$

$$\text{But } \sin^{-1}\left(\frac{5}{13}\right) = A \text{ and } \sin^{-1}\left(\frac{4}{5}\right) = B$$

$$\sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \sin^{-1}\left(\frac{63}{65}\right) \text{ Hence shown.}$$

### SOLVING EQUATIONS INVOLVING INVERSES OF TRIGONOMETRIC FUNCTIONS

#### Example 70

$$\text{Solve for } x \text{ from } \cos^{-1} x + \cos^{-1}(x\sqrt{2}) = \frac{\pi}{2}$$

#### Solution

$$\cos^{-1} x + \cos^{-1}(x\sqrt{2}) = \frac{\pi}{2}$$

$$\text{Let } A = \cos^{-1} x \Rightarrow \cos A = x \text{ and } B = \cos^{-1}(x\sqrt{2}) \Rightarrow \cos B = x\sqrt{2}$$

$$\text{Then } A+B = \frac{\pi}{2}$$

$$\cos(A+B) = \cos\left(\frac{\pi}{2}\right)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B = 0$$

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$$\cos A \cos B - \left( \sqrt{1 - \cos^2 A} \right) \left( \sqrt{1 - \cos^2 B} \right) = 0$$

$$(x)(x\sqrt{2}) - \left( \sqrt{1 - x^2} \right) \left( \sqrt{1 - 2x^2} \right) = 0$$

$$\sqrt{2}x^2 - \sqrt{(1 - x^2)(1 - 2x^2)} = 0$$

$$\sqrt{2}x^2 = \sqrt{(1 - x^2)(1 - 2x^2)}$$

$$2x^4 = (1 - x^2)(1 - 2x^2)$$

$$2x^4 = 1 - 2x^2 - x^2 + 2x^4$$

$$3x^2 - 1 = 0$$

$$x = \pm \sqrt{\frac{1}{3}} = \pm \frac{\sqrt{3}}{3}$$

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### Example 71

Solve for  $x$  from  $\tan^{-1} 2x + \tan^{-1} 4x = \frac{\pi}{2}$

#### Solution

$$\tan^{-1} 2x + \tan^{-1} 4x = \frac{\pi}{2}$$

Let  $A = \tan^{-1} 2x \Rightarrow \tan A = 2x$  and  $B = \tan^{-1} 4x \Rightarrow \tan B = 4x$

$$A + B = \frac{\pi}{2}$$

$$\tan(A + B) = \tan\left(\frac{\pi}{2}\right)$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \sqrt{3}$$

$$\frac{2x + 4x}{1 - (2x)(4x)} = \sqrt{3}$$

$$\frac{6x}{1 - 8x^2} = \sqrt{3}$$

$$6x = \sqrt{3} - 8\sqrt{3}x^2$$

$$8\sqrt{3}x^2 + 6x - \sqrt{3} = 0$$

The solution is  $x = 0.198$  or  $x = -0.631$

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**MISCELLANEOUS EXERCISE**

1. Draw the graph of  $y = 2 + 4\cos x$  and state domain and range of  $y$ .
2. Evaluate the following

(a)  $\lim_{\theta \rightarrow 0} \left( \frac{\cos 2\theta - 1}{\theta \sin 5\theta} \right)$

(b)  $\lim_{\theta \rightarrow 0} \left( \frac{\sin 3\theta + \tan 5\theta}{2\theta} \right)$

(c)  $\lim_{\theta \rightarrow 0} \left( \frac{21 + 7 \tan \theta - 20 \cos \theta}{1 + \sin 2\theta} \right)$

(d)  $\lim_{\theta \rightarrow 0} \left( \frac{4\theta}{\sin \theta} \right)$

(e)  $\lim_{\theta \rightarrow 0} \left( \frac{\tan 2\theta}{\sin \theta} \right)$

(f)  $\lim_{\theta \rightarrow 0} \left( \frac{\sin \theta}{1 - \cos 2\theta} \right)$

(g)  $\lim_{\theta \rightarrow 0} \left( \frac{\sin 3\theta + \tan \theta}{\cos 2\theta} \right)$

(h)  $\lim_{\theta \rightarrow 0} \sin(\theta + 45^\circ)$

(i)  $\lim_{\theta \rightarrow 0} \cos(\theta + 30^\circ)$

(j)  $\lim_{\theta \rightarrow 0} \left( \frac{1 + \sin \theta}{5 + 3 \tan \theta - 3 \cos \theta} \right)$

3. Prove the following

(a)  $\cos^{-1}\left(\frac{15}{17}\right) - \cos^{-1}\left(\frac{7}{25}\right) = \cos^{-1}\left(\frac{297}{425}\right)$

(b)  $\sin^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{1}{3}\right) = \sin^{-1}\left(\frac{1 - 2\sqrt{6}}{6}\right)$

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- (c)  $\tan^{-1}\left(\frac{3}{4}\right) + \cot^{-1}\left(\frac{15}{8}\right) = \tan^{-1}\left(\frac{77}{36}\right)$
- (d)  $\sin^{-1}\left(-\frac{3}{5}\right) - \cos^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{63}{16}\right)$
- (e)  $2\sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \tan^{-1}\left(-\frac{253}{204}\right)$
- (f)  $\tan^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{7}{5}\right) = \tan^{-1}\left(12\frac{1}{2}\right)$
- (g)  $2\sin^{-1}\left(\frac{4}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{323}{325}\right)$

4. Solve the following

- (a)  $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = 45^\circ$
- (b)  $\sin^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$
- (c)  $\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$
- (d)  $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$
- (e)  $\cos^{-1}\left(\frac{12}{13}\right) + \tan^{-1}\left(\frac{1}{4}\right) = \frac{\pi}{3}$

5. Find the general solution of the following

- (a)  $\cos 2x - 3\sin x + 1 = 0$
- (b)  $3\sin x + 4\cos x = 2$
- (c)  $\sin x \cos x = \frac{1}{2}$
- (d)  $\sin\left(\frac{1}{2}x\right) + \cos x = 1$
- (e)  $(\tan x - 1)(2\sin x + 1) = 0$

6. Prove the following

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$$(a) \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos 8x}}} = 2\cos x$$

$$(b) \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$(c) \csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

$$(d) \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$$

$$(e) \frac{\cot^2 x - 1}{1 + \cot^2 x} = 1 - 2\sin^2 x$$

$$(f) \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \csc x - \sec x$$

$$(g) \frac{\cot A}{\csc A - 1} = \frac{\csc A - 1}{\cot A}$$

7. If  $A + B + C = \pi$  show that

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$$

8. Simplify

$$(a) \frac{\tan 15^\circ + \tan 30^\circ}{1 - \tan 15^\circ \tan 30^\circ}$$

$$(b) \sin 40^\circ \cos 50^\circ + \sin 50^\circ \cos 40^\circ$$

9. Solve for  $x$ :  $\sin(x + 30^\circ) = \cos(x + 30^\circ)$  from  $0^\circ$  to  $360^\circ$  inclusive

10. Prove that

$$(a) \tan x = \frac{3\sin x + \sin 2x}{1 + 3\cos x + \cos 2x}$$

$$(b) \frac{\cos 2A}{\cos A - \sin A} = \cos A + \sin A$$

11. Solve for  $\theta$  between  $-180^\circ \leq \theta \leq 180^\circ$  from  $\sqrt{3} \tan 6\theta = 1$

12. Draw the graph of

$$(a) y = \sin x \text{ from } 0^\circ \leq x \leq 180^\circ$$

$$(b) y = \tan x \text{ from } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

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13. Without using the table find the value of

- (a)  $\sin 18^\circ$ ,
- (b)  $\cos 18^\circ$
- (c)  $\tan 18^\circ$

Hint: let  $x = 18^\circ$ ,  $2x = 36^\circ$  then  $5x = 90^\circ$ ]

14. Prove the identity

- (a)  $\sin \theta + 2\sin 3\theta + \sin 5\theta = 2\cos \theta (\sin 2\theta + \sin 4\theta)$
- (b) Given that  $3\tan \theta - \sec \theta = 1$  find the possible value of  $3\sec \theta + \tan \theta$

15. If  $2x + y = 45^\circ$  show that  $y = \tan^{-1} \left( \frac{1 - 2\tan x - \tan^2 x}{1 + 2\tan x - \tan^2 x} \right)$

16. Evaluate  $\lim_{x \rightarrow 0} \left( x \sin \left( \frac{1}{x} \right) \right)$

17. Simplify

- (a)  $\sin(\sin^{-1} x + \cos^{-1} x)$
- (b)  $\tan \left( \cos^{-1} \left( \frac{1}{2} \right) - \tan^{-1} \left( \frac{\sqrt{3}}{2} \right) \right)$

18. Use t substitution, to solve  $3\sin x + 4\cos x - 2 = 0$  for  $x$  between 0 and  $2\pi$  inclusive.

19. Prove that for any triangle

- (a)  $\sin \frac{1}{2}(B - C) = \frac{b - c}{a} \cos \frac{1}{2} A$
- (b)  $b + c = a \cos \frac{1}{2}(B - C) \csc \frac{1}{2} A$ , where  $a$ ,  $b$  and  $c$  are sides of the triangle.

20. Prove that

- (a)  $\frac{\sec A - \csc A}{\tan A - \cot A} = \frac{\tan A + \cot A}{\sec A + \csc A}$
- (b)  $\frac{\cos A}{\sin A} - \frac{\sin A}{\cos A} = \frac{2\cos(A + B)}{\sin 2B}$

21. Solve the trigonometric equation  $\sec^2 \theta + \tan \theta - 1 = 0$  for  $0 \leq \theta \leq 2\pi$

22. Factorize  $\cos \theta - \cos 3\theta - \cos 5\theta + \cos 7\theta$

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23. Prove that  $\frac{\cos^2 \theta - 3\cos\theta + 2}{\sin^2 \theta} = \frac{2 - \cos\theta}{1 + \cos\theta}$

24. Show that  $\frac{\sin 3A \sin 6A + \sin A \sin 2A}{\sin 3A \cos 6A + \sin A \cos 2A} = \tan 5A$

25. Show that  $\frac{\sin^2 \theta + 2\sin\theta + 1}{\cos^2 \theta} = \frac{\cos^2 \theta}{1 - \sin\theta + \sin^2 \theta}$

26. (a) Prove that  $\tan \theta = \frac{\sin 2\theta}{1 + \cos 2\theta}$  and hence obtain  $\tan 15^\circ$  and  $\tan 67\frac{1}{2}^\circ$

as surds, in their simplest forms.

(b) Prove that  $\sin \frac{1}{2} A - \cos \frac{1}{2} A = \pm \sqrt{1 - \sin A}$  determining for what range of angles  $A$  between 0 and  $4\pi$  the positive sign is to be taken.

27. A line  $AP$  is drawn through the vertex  $A$  of a triangle  $ABC$ , so that  $P$  and  $C$  are on opposite sides of  $AB$ , and angle  $BAP = \theta$ . By projecting the sides of the triangle  $ABC$  on  $AP$ , prove that

$$c \cos \theta = a \cos(B - \theta) + b \cos(A + \theta).$$

Deduce from this result the two formulae,  $c = a \cos B + b \cos A$  and  $a \sin B = b \sin A$ . Further, by considering the special case  $\theta = A = B$ , deduce that

$$\cos 2A = 2 \cos^2 A - 1 \quad \text{A.E.B}$$

28. (a) Find the solution between  $0^\circ$  and  $360^\circ$  inclusive of the equations

i)  $\cot \frac{3x}{2} = \tan 171^\circ$

ii)  $\sin 3x + 3 \cos 2x + \sin x = 3$

(b) Find the solution in radians between 0 and  $2\pi$  of the equation

$$\cos 3x = \sin x$$

29. From a point  $A$  of a straight level road which runs due north, a peak bears due west. A man walks along the road, starting from  $A$  and going at a steady speed. After 10 min. the elevation of the peak is  $\alpha$ ; after a further 20 min. it is  $\beta$ . If the elevation of the peak from  $A$  is  $\theta$ , prove that

$$8 \cot^2 \theta = 9 \cot^2 \alpha - \cot^2 \beta.$$

If the man's speed is 4m per hour and  $\alpha$ ,

$\beta$  are  $17^\circ 18'$  and  $10^\circ 24'$  respectively, calculate the height of the peak above the level of the road, to the nearest 50 m.

30. (a) Show that  $\tan A = 3$  if  $\sin A + \cos A = 2(\sin A - \cos A)$ , hence solve  $\sin A + \cos A - 2(\sin A - \cos A) = 0$  for  $0^\circ \leq A \leq 360^\circ$

(b) A function  $f$  is defined by  $f(x) = 3 - 2 \sin x$  for  $0^\circ \leq x \leq 360^\circ$  find

(i) The range of  $f(x)$

(ii) Sketch the graph of  $f(x)$

31. When a motor-cyclist is travelling along a straight stretch of road from south to north at a steady speed of 30 km per hour, the wind appears to him to come from a direction  $N40^\circ E$ . When he returns along the same road at the same speed the wind appears to come from a direction  $S30^\circ E$ . Find, by drawing or otherwise the true magnitude and direction of the wind.

32. Prove that  $\sin 3A - \cos 3A = (\sin A + \cos A)(2 \sin 2A - 1)$ , hence or otherwise, find the value of  $A$  such that  $0^\circ \leq A \leq 180^\circ$  for which  $3(\sin 3A - \cos 3A) = 2(\sin A + \cos A)$ .

33. In the quadrilateral  $ABCD$ ,  $AB = 13\text{cm}$ ,  $BC = 20\text{cm}$ ,  $CD = 48\text{cm}$  and  $\angle BAC = \angle DBC$ . Without using tables

(a) Show that  $\cos BAC = \frac{5}{13}$

(b) Show that  $\cos ACB = \frac{4}{5}$

(c) Show that  $AB = AD$

(d) Calculate the area of the quadrilateral  $ABCD$ .

34. (a) Express  $3\cos\theta + 2\sin\theta$  in the form  $R\cos(\theta - A)$ , where  $R > 0$  and  $0^\circ < A < 90^\circ$ , state the value of  $R$

(b) Solve the equation  $3\cos\theta + 2\sin\theta = 3.5$  giving your answer in the interval  $0^\circ \leq \theta \leq 180^\circ$

(c) Sketch the graph of  $y = 3\cos\theta + 2\sin\theta$  for  $0^\circ \leq \theta \leq 180^\circ$

35. (a) From a point  $P$  in a horizontal plane a man observes the summit  $S$  of a mountain to bear due north at an elevation  $\theta$ . When the man has walked a distance  $2a$  on a bearing  $\alpha$  east of north to a point  $Q$  in the horizontal

plane he observes that the elevation of S from Q is again  $\theta$ . If  $h$  is the height of S above the horizontal plane containing P and Q show that  $h = a \tan \theta \sec \alpha$

- (c) When the man has walked a further distance  $a$  in the same plane and in the same direction to a point R he observes that the elevation of S from R is  $\phi$ .

Show that  $\cot^2 \phi = (3 \cos^2 \alpha + 1) \cot^2 \theta$  and that the distance RS is  $\sqrt{a(\sec^3 \alpha \sec^2 \theta + 3)}$

36. (a) Solve the equation  $2 \sec^2 x - \tan x = 5$  for  $0^\circ \leq x \leq 360^\circ$

(b) Show that  $\sin x \tan x = \frac{1 - \cos^2 x}{\cos x}$  hence or otherwise solve

$$2 \sin x \tan x = 3 \text{ for } 0^\circ \leq x \leq 360^\circ$$

37. Prove that  $\sin^2 A \cos^2 A = \frac{1}{8}(1 - \cos 4A)$  hence find  $\int \sin^2 \theta \cos^2 \theta d\theta$

38. Find all real values of  $x$  and  $y$  that satisfy the simultaneous equations

$$\begin{cases} 16 \sin x + 9 \tan y = 35 \\ 64 \sin^2 x + 27 \tan^2 y = 259 \end{cases} \text{ where } x, y \text{ are the measures of the angles expressed in degrees.}$$

39. Prove that  $\cot A - \cot 2A = \operatorname{cosec} 2A$

40. Prove that  $2 \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + 2 \tan^{-1}\left(\frac{1}{8}\right) = \frac{1}{4} \pi$

41. (a) If  $\tan(45^\circ + \theta) - \tan(45^\circ - \theta) = 2 \tan 2\theta$  (b) Prove that

$$\frac{1 + \tan^2 x}{1 + \cot^2 x} = \left( \frac{1 - \tan x}{1 - \cot x} \right)^2$$

42. (a) Prove that

$$4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \frac{\pi}{4}$$

- (b) Using the expansion  $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots$  evaluate  $\pi$  to five decimal places.

43. Given that  $\cos x = \frac{\sqrt{3}}{2}$  and  $0^\circ \leq x \leq 360^\circ$ . Find without using calculator or table the value of  $\tan x$  and  $\sin x$

44. (a) Prove that

$$4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right) = \frac{\pi}{4}$$

- (b) Using the expansion  $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots$  evaluate  $\pi$  to five decimal places.

45. Let  $x = \sin \theta$  or otherwise, show that  $\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} = -i$

46. (a) Solve for  $x$  if  $4 \sin x \cos x + 2 \sin x + 2 \cos x + 1 = 0$

- (b) Find the least difference between the roots, in the first quadrant of the equation  $4 \cos x (2 - 3 \sin^2 x) + (\cos 2x + 1) = 0$

- (c) Simplify  $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1-\sin x} - \sqrt{1-\sin x}}\right)$

- (d) Given  $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = 180^\circ$  show that

$$x^4 + y^4 + z^4 + 4(xyz)^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$$

- (e) By expressing  $\cos A + 2 \sin A$  in the form of  $Q \sin(A+B)$  where  $B$  is an acute angle, find the minimum and maximum values of the expression and hence find the values of  $A$  and  $B$ .

- (f) Show that  $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}$

hence show that  $\tan A + \tan B + \tan C = \tan A \tan B \tan C$  if  $A, B$  and  $C$  are angles of the triangles.

47. (a) If  $2A = \arcsin(3/4)$ , without using a calculator show that

$$\sin A - \cos A = \pm \frac{1}{2}$$

- (b) Prove the following identities

$$\text{i) } \sqrt{\frac{1+\cos \theta}{1-\cos \theta}} = \operatorname{cosec} \theta + \cot \theta$$

- ii)  $\frac{1 + \cos A - \sin^2 A}{\sin A(1 + \cos A)} = \cot A$
- (c) Express  $2\cos\theta + 3\sin\theta$  in the form of  $R\sin(\theta + \alpha)$  then find
- Maximum and minimum values
  - The value of  $\theta$  for which the maximum and minimum values occur.
- (d) If  $p = \cos A - \cos B$  and  $q = \sin A - \sin B$ , express  $\cos(A - B)$  and  $\sin(A + B)$  in terms of  $p$  and  $q$ .

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# TRIGONOMETRY

# 7

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