

The correct solutions to the DE

$$\frac{d^2y}{dx^2} - \frac{1}{x}\left(\frac{dy}{dx}\right) + \frac{1-x}{x}y = 4xe^x$$

are **EITHER**

$$y = x^2e^x - \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right); c_1 \in R$$

OR

$$y = (x^2 + c)e^x - \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right); c, c_1 \in R$$

VERIFICATIONS

TASK-1: Show that

$$y = x^2e^x - \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right); c_1 \in R$$

is a solution to the DE

$$\frac{d^2y}{dx^2} - \frac{1}{x}\left(\frac{dy}{dx}\right) + \frac{1-x}{x}y = 4xe^x$$

SOLUTION:

$$\begin{aligned} y &= x^2e^x - \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right) \Rightarrow y + \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right) = x^2e^x \\ y &= x^2e^x - \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right) \\ \Rightarrow \frac{dy}{dx} &= 2xe^x + x^2e^x + \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right) - \frac{1}{2}c_1e^{-x} \\ &= 2xe^x + y + \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right) + \frac{1}{2}c_1e^{-x}\left(x + \frac{1}{2}\right) - \frac{1}{2}c_1e^{-x} \\ &= 2xe^x + y + c_1e^{-x}\left(x + \frac{1}{2}\right) - \frac{1}{2}c_1e^{-x} = 2xe^x + y + c_1xe^{-x} \\ \frac{dy}{dx} &= y + 2xe^x + c_1xe^{-x} \Rightarrow y = \frac{dy}{dx} - 2xe^x - c_1xe^{-x} \\ \frac{dy}{dx} &= y + 2xe^x + c_1xe^{-x} \Rightarrow \frac{1}{x}\left(\frac{dy}{dx}\right) = \frac{y}{x} + 2e^x + c_1e^{-x} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{dy}{dx} + 2e^x + 2xe^x + c_1e^{-x} - c_1xe^{-x} \\ \Rightarrow \frac{d^2y}{dx^2} - \frac{1}{x}\left(\frac{dy}{dx}\right) &= \frac{dy}{dx} + 2e^x + 2xe^x + c_1e^{-x} - c_1xe^{-x} - \left(\frac{y}{x} + 2e^x + c_1e^{-x}\right) \\ &= \frac{dy}{dx} - \frac{y}{x} + 2xe^x - c_1xe^{-x} \\ \frac{d^2y}{dx^2} - \frac{1}{x}\left(\frac{dy}{dx}\right) &= \left(\frac{dy}{dx}\right) - \frac{y}{x} + 2xe^x - c_1xe^{-x} \end{aligned}$$

But we know from above that

$$\frac{dy}{dx} = y + 2xe^x + c_1xe^{-x}$$

$$\begin{aligned}
\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{x} \left(\frac{dy}{dx} \right) &= (y + 2xe^x + c_1xe^{-x}) - \frac{y}{x} + 2xe^x - c_1xe^{-x} \\
\frac{d^2y}{dx^2} - \frac{1}{x} \left(\frac{dy}{dx} \right) &= y - \frac{y}{x} + 4xe^x = \left(\frac{x-1}{x} \right) y + 4xe^x \\
\frac{d^2y}{dx^2} - \frac{1}{x} \left(\frac{dy}{dx} \right) - \left(\frac{x-1}{x} \right) y &= 4xe^x \\
\Rightarrow \frac{d^2y}{dx^2} - \frac{1}{x} \left(\frac{dy}{dx} \right) + \left(\frac{1-x}{x} \right) y &= 4xe^x
\end{aligned}$$

The above steps therefore verify also that

$$y = x^2e^x - \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right); c, c_1 \in R$$

is a solution to the DE

$$\frac{d^2y}{dx^2} - \frac{1}{x} \left(\frac{dy}{dx} \right) + \left(\frac{1-x}{x} \right) y = 4xe^x$$

TASK-2: Show that

$$y = (x^2 + c)e^x - \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right); c, c_1 \in R$$

Is a solution to the DE

$$\frac{d^2y}{dx^2} - \frac{1}{x} \left(\frac{dy}{dx} \right) + \frac{1-x}{x} y = 4xe^x$$

SOLUTION:

$$\begin{aligned}
y &= (x^2 + c)e^x - \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right) \Rightarrow y + \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right) = (x^2 + c)e^x \\
y &= (x^2 + c)e^x - \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right) \\
\Rightarrow \frac{dy}{dx} &= 2xe^x + (x^2 + c)e^x + \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right) - \frac{1}{2}c_1e^{-x} \\
&= 2xe^x + y + \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right) + \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right) - \frac{1}{2}c_1e^{-x} \\
&= 2xe^x + y + c_1e^{-x} \left(x + \frac{1}{2} \right) - \frac{1}{2}c_1e^{-x} = 2xe^x + y + c_1xe^{-x} \\
&\Rightarrow \frac{dy}{dx} = y + 2xe^x + c_1xe^{-x}
\end{aligned}$$

At this stage, the other constant c vanishes from the expressions and from this point onwards, the rest of the steps become identical to those presented in TASK-1, which finally lead again to the conclusion that

$$y = (x^2 + c)e^x - \frac{1}{2}c_1e^{-x} \left(x + \frac{1}{2} \right)$$

is a solution to the DE

$$\frac{d^2y}{dx^2} - \frac{1}{x} \left(\frac{dy}{dx} \right) + \frac{1-x}{x} y = 4xe^x$$