2 Example 2.

Find the area of the region R bounded by $y^2 = a^2 - ax, y = a + x.$

SOLUTION: (ADD A SKETCH)

The area A is given by

$$A = \iint \quad dxdy$$

Looking at the above graphical sketch, we easily observe that there two regions lieng on each side of the vertical axis

For the region on the left side, as we move vertically from bottom to top, we first meet the parabola $y^2 = a^2 - ax \Longrightarrow y = -\sqrt{a^2 - ax}$ (lower limit of integration for the variable y) and then the line $y = a + x$ (upper limit of integration of the same variable y); whereas the values of x are in the range $-3a \leq x \leq 0$ within the region

In this case therefore

$$A_1 = \int_{-3a}^{0} \left(\int_{-\sqrt{a^2-ax}}^{a+x} dy = \right) dx$$

On considering the region lieing on the right hand side of the same (vertical) axis, and while moving vertically from top to bottom, we again first meet the parabola $y^2 = a^2 - ax \Longrightarrow y = -\sqrt{a^2 - ax}$ (lower limit of integration for the variable y) and then the same parabola $y^2 = a^2 - ax \Longrightarrow y = \sqrt{a^2 - ax}$ at the top (upper limit of integration of the same variable y) with x values being in the range $0 \leq x \leq a$

Therefore

$$A_2 = \int_{0}^{a} \left(\int_{-\sqrt{a^2-ax}}^{\sqrt{a^2-ax}} dy = \right) dx$$

Finally;

$$A = \iint \quad dxdy = A_1 + A_2$$

$$\Longrightarrow A = \int_{-3a}^{0} \left(\int_{-\sqrt{a^2-ax}}^{a+x} dy = \right) dx + \int_{0}^{a} \left(\int_{-\sqrt{a^2-ax}}^{\sqrt{a^2-ax}} dy = \right) dx = I_k + I_m$$

where

$$I_k = \int_{-3a}^{0} \left(\int_{-\sqrt{a^2-ax}}^{a+x} dy = \right) dx;$$

$$I_m = \int_{0}^{a} \left(\int_{-\sqrt{a^2-ax}}^{\sqrt{a^2-ax}} dy = \right) dx$$

Let's first begin with I_k

$$I_k = \int_{-3a}^{0} \left(\int_{-\sqrt{a^2-ax}}^{a+x} dy = \right) dx = \int_{-3a}^{0} \left(a + x + \sqrt{a^2 - ax}\right) dx$$

Again, let

$$I_p = \int \left(\sqrt{a^2 - ax}\right) dx$$

and

$$u = \sqrt{a^2 - ax} \Longrightarrow u^2 = a^2 - ax$$

$$\Longrightarrow 2udu = -adx \Longrightarrow dx = -\frac{2}{a}udu$$

$$\Longrightarrow \int \left(\sqrt{a^2 - ax}\right) dx = \int u\left(-\frac{2}{a}udu\right) = -\frac{2}{a}\int u^2 du = -\frac{2}{a}\left(\frac{u^3}{3}\right)$$

$$\Longrightarrow \int \left(\sqrt{a^2 - ax}\right) dx = -\frac{2}{3a}\left(\sqrt{a^2 - ax}\right)^3$$

Therefore

$$\int_{-3a}^{0} \left(a + x + \sqrt{a^2 - ax}\right) dx = \left(ax + \frac{x^2}{2} - \frac{2}{3a}\left(\sqrt{a^2 - ax}\right)^3\right)\Big|_{-3a}^{0}$$

$$= -\frac{2}{3}a^2 - \left[a(-3a) + \frac{(-3a)^2}{2} - \frac{2}{3a}\left(\sqrt{a^2 - a(-3a)}\right)^3\right]$$

$$= -\frac{2}{3}a^2 - \left[-3a^2 + \frac{9}{2}a^2 - \frac{2}{3a}\left(\sqrt{4a^2}\right)^3\right]$$

$$-\frac{2}{3}a^2 - \frac{3}{2}a^2 + \frac{16}{3}a^2 = \left(\frac{-4 - 9 + 32}{6}\right)a^2 = \frac{19}{6}a^2$$

$$\Longrightarrow I_k = \frac{19}{6}a^2$$

Again, going back to our second integral I_m we have

$$I_m = \int_{0}^{a} \left(\int_{-\sqrt{a^2-ax}}^{\sqrt{a^2-ax}} dy = \right) dx = 2\int_{0}^{a} \left(\sqrt{a^2 - ax} =\right) dx = 2\left(-\frac{2}{3a}\left(\sqrt{a^2 - ax}\right)^3\right)\Big|_{0}^{a}$$

$$= -\frac{4}{3a}(0 - a^3) = \frac{4}{3}a^2$$

$$\Longrightarrow I_m = \frac{4}{3}a^2$$

Eventually

$$I_k + I_m = \frac{19}{6}a^2 + \frac{4}{3}a^2 = \frac{19 + 8}{6}a^2 = \frac{27}{6}a^2 = \frac{9}{2}a^2$$

Hence

$$\int_{-3a}^{0}\left(\int_{-\sqrt{a^2-ax}}^{a+x} dy\right)dx+\int_{0}^{a}\left(\int_{-\sqrt{a^2-ax}}^{\sqrt{a^2-ax}} dy\right)dx=\frac{9}{2}a^2\ square\ units$$