LAPLACE'S EQUATION IN SPHERICAL POLAR CO-ORDINATES

$$\nabla_s^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right] + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

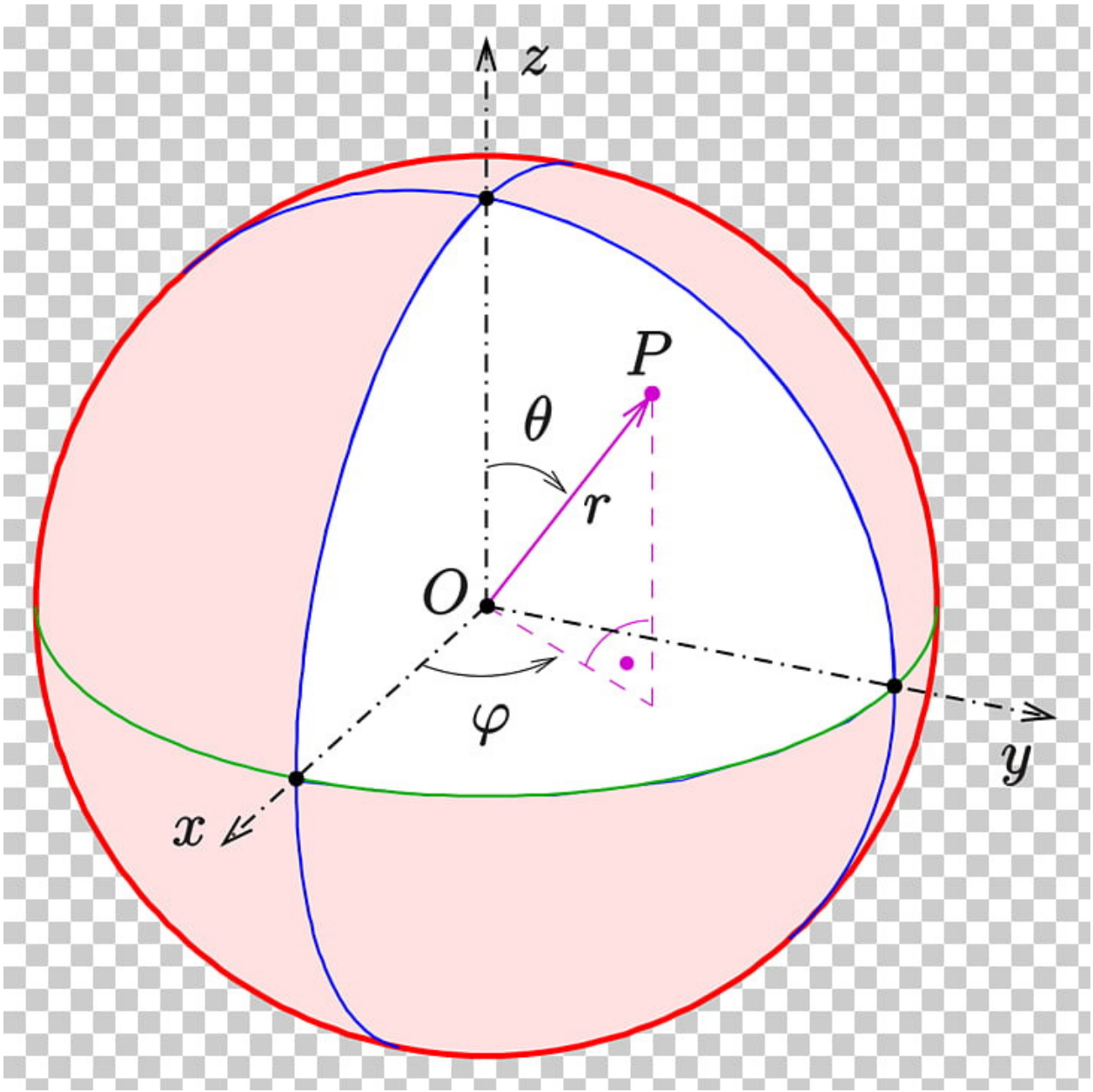
$$\nabla_s^2 V = \frac{1}{r^2}\left[\frac{\partial V}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)\right] + \frac{1}{r^2 sin\theta}\frac{\partial V}{\partial \theta}\left(sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2}$$

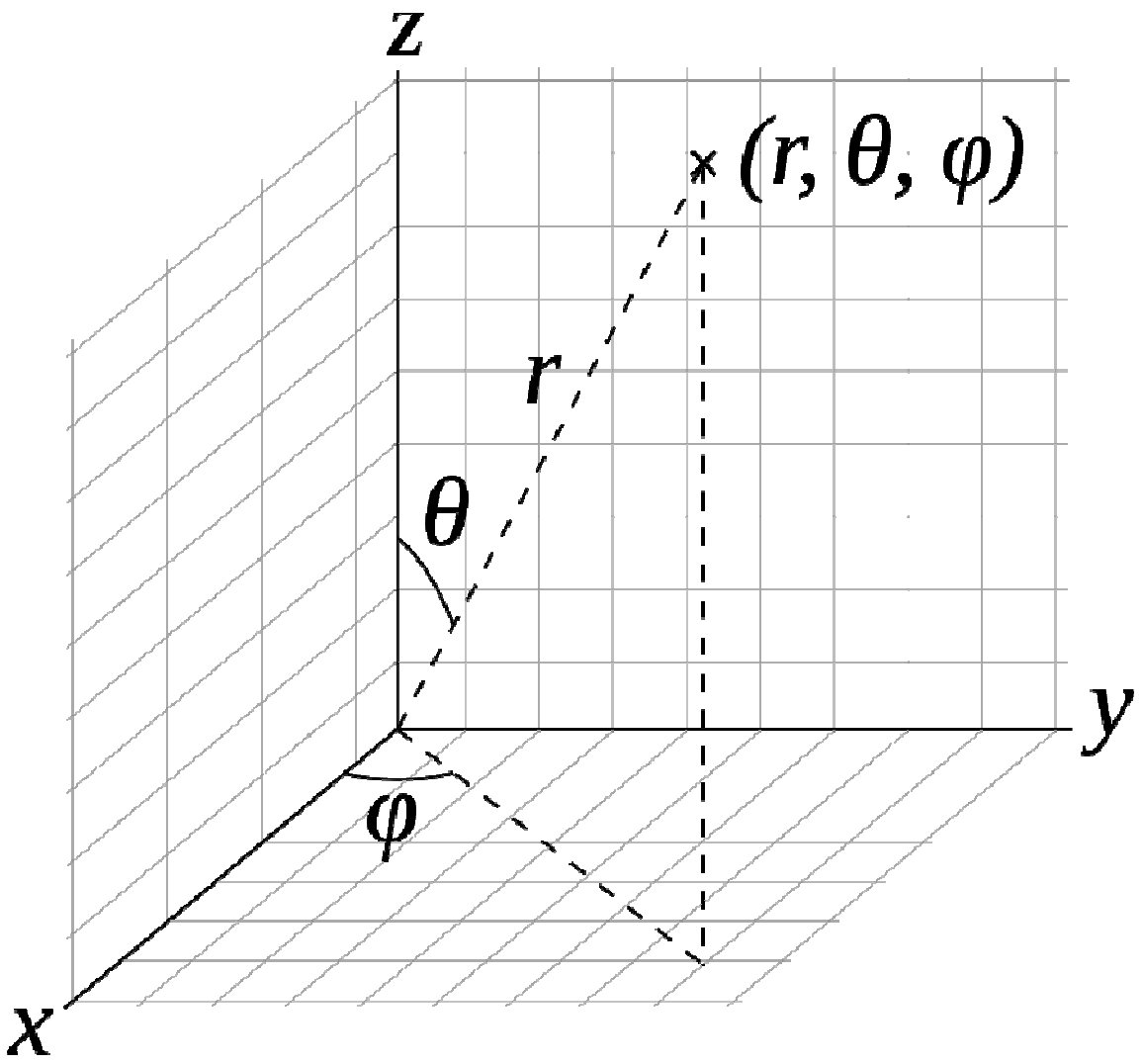
Transformation of coordinates from Cartesian to Spherical, of a given function let's say Ψ, (that is $\Psi(x, y, z) \rightarrow \Psi(r, \varphi, \theta)$) is governed by the following fundamental relationships:

$$x = rcos\varphi sin\theta$$

$$y = rsin\varphi sin\theta$$

$$z = rcos\theta$$


z
P
θ
r
O
φ
y
x



Then, the partial derivatives of the Cartesian coordinates with respect to the Spherical ones are:

$$\frac{\partial x}{\partial r} = cos\varphi sin\theta;\ \frac{\partial y}{\partial r} = sin\varphi sin\theta;\ \frac{\partial z}{\partial r} = cos\theta$$

$$\frac{\partial x}{\partial \theta} = rcos\varphi cos\theta;\ \frac{\partial y}{\partial \theta} = rsin\varphi cos\theta;\ \frac{\partial z}{\partial \theta} = -rsin\theta$$

$$\frac{\partial x}{\partial \varphi} = -rsin\varphi sin\theta;\ \frac{\partial y}{\partial \varphi} = rcos\varphi sin\theta;\ \frac{\partial z}{\partial \varphi} = 0$$

Now bearing in mind that for any function Ψ;

$$\frac{\partial \Psi}{\partial r} = \frac{\partial x}{\partial r}\cdot\frac{\partial \Psi}{\partial x} + \frac{\partial y}{\partial r}\cdot\frac{\partial \Psi}{\partial y} + \frac{\partial z}{\partial r}\cdot\frac{\partial \Psi}{\partial z}$$

then

$$\frac{\partial \Psi}{\partial r} = cos\varphi sin\theta \frac{\partial \Psi}{\partial x} + sin\varphi sin\theta \frac{\partial \Psi}{\partial y} + cos\theta \frac{\partial \Psi}{\partial z} \Longrightarrow \frac{\partial}{\partial r}$$

$$= cos\varphi sin\theta \frac{\partial}{\partial x} + sin\varphi sin\theta \frac{\partial}{\partial y} + cos\theta \frac{\partial}{\partial z}$$

Similarly

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}.\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}.\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}.\frac{\partial}{\partial z} \Longrightarrow \frac{\partial}{\partial \theta} = rcos\varphi cos\theta \frac{\partial}{\partial x} + rsin\varphi cos\theta \frac{\partial}{\partial y} - rsin\theta \frac{\partial}{\partial z}$$

and finally;

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}.\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}.\frac{\partial}{\partial y} + \frac{\partial z}{\partial \varphi}.\frac{\partial}{\partial z} \Longrightarrow \frac{\partial}{\partial \varphi} = -rsin\varphi sin\theta \frac{\partial}{\partial x} + rcos\varphi sin\theta \frac{\partial}{\partial y} + 0$$

From these relationship, we finally observe the facts that

$$\frac{\partial}{\partial r} = cos\varphi sin\theta \frac{\partial}{\partial x} + sin\varphi sin\theta \frac{\partial}{\partial y} + cos\theta \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \theta} = rcos\varphi cos\theta \frac{\partial}{\partial x} + rsin\varphi cos\theta \frac{\partial}{\partial y} - rsin\theta \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial \varphi} = -rsin\varphi sin\theta \frac{\partial}{\partial x} + rcos\varphi sin\theta \frac{\partial}{\partial y}$$

In matrix notation, these relationship can be presented as follows:

$$\begin{pmatrix} cos\varphi sin\theta & sin\varphi sin\theta & cos\theta \\ cos\varphi cos\theta & sin\varphi cos\theta & -sin\theta \\ -sin\varphi sin\theta & cos\varphi sin\theta & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r}\frac{\partial}{\partial \varphi} \end{pmatrix}$$

Now, we need to determine the values (or expressions) of partial derivates in Cartesian coordinates from the above 3x3 matrix, in terms of the spherical polar ones

For simplicity in working with the partial derivatives operators, let

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} cos\varphi sin\theta & sin\varphi sin\theta & cos\theta \\ cos\varphi cos\theta & sin\varphi cos\theta & -sin\theta \\ -sin\varphi sin\theta & cos\varphi sin\theta & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \varphi} \end{pmatrix}$$

From this matrix, we therefore need to solve for a, b and c.

Let

$$A = \begin{pmatrix} cos\varphi sin\theta & sin\varphi sin\theta & cos\theta \\ cos\varphi cos\theta & sin\varphi cos\theta & -sin\theta \\ -sin\varphi sin\theta & cos\varphi sin\theta & 0 \end{pmatrix}$$

We therefore need to find first, the inverse of the matrix A

The transpose of this matrix is

$$A^T = \begin{pmatrix} cos\varphi sin\theta & cos\varphi cos\theta & -sin\varphi sin\theta \\ sin\varphi sin\theta & sin\varphi cos\theta & cos\varphi sin\theta \\ cos\theta & -sin\theta & 0 \end{pmatrix}$$

The co-factors of A^T are as follows:

$$\begin{pmatrix} sin\varphi cos\theta & cos\varphi sin\theta \\ -sin\theta & 0 \end{pmatrix} \begin{pmatrix} sin\varphi sin\theta & cos\varphi sin\theta \\ cos\theta & 0 \end{pmatrix} \begin{pmatrix} sin\varphi sin\theta & sin\varphi cos\theta \\ cos\theta & -sin\theta \end{pmatrix}$$

$$\begin{pmatrix} cos\varphi cos\theta & -sin\varphi sin\theta \\ -sin\theta & 0 \end{pmatrix} \begin{pmatrix} cos\varphi sin\theta & -sin\varphi sin\theta \\ cos\theta & 0 \end{pmatrix} \begin{pmatrix} cos\varphi sin\theta & cos\varphi cos\theta \\ cos\theta & -sin\theta \end{pmatrix}$$

$$\begin{pmatrix} cos\varphi cos\theta & -sin\varphi sin\theta \\ sin\varphi cos\theta & cos\varphi sin\theta \end{pmatrix} \begin{pmatrix} cos\varphi sin\theta & -sin\varphi sin\theta \\ sin\varphi sin\theta & cos\varphi sin\theta \end{pmatrix} \begin{pmatrix} cos\varphi sin\theta & cos\varphi cos\theta \\ sin\varphi sin\theta & sin\varphi cos\theta \end{pmatrix}$$

C

$$= \begin{pmatrix} cos\varphi\, sin^2\theta & -cos\varphi cos\theta sin\theta & -sin\varphi\, sin^2\theta - sin\varphi\, cos^2\theta \\ -sin\varphi\, sin^2\theta & sin\varphi cos\theta sin\theta & -cos\varphi\, sin^2\theta - cos\varphi\, cos^2\theta \\ cos^2\varphi\, cos\theta sin\theta + sin^2\varphi\, cos\theta sin\theta & cos^2\varphi\, sin^2\theta + sin^2\varphi\, sin^2\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} cos\varphi \sin^2\theta & -cos\varphi cos\theta sin\theta & -sin\varphi\,(\sin^2\theta + \cos^2\theta) \\ -sin\varphi \sin^2\theta & sin\varphi cos\theta sin\theta & -cos\varphi(\sin^2\theta + \cos^2\theta) \\ cos\theta sin\theta & \sin^2\theta & 0 \end{pmatrix}$$

$$\Longrightarrow C = \begin{pmatrix} cos\varphi \sin^2\theta & -cos\varphi cos\theta sin\theta & -sin\varphi \\ -sin\varphi \sin^2\theta & sin\varphi cos\theta sin\theta & -cos\varphi \\ cos\theta sin\theta & \sin^2\theta & 0 \end{pmatrix}$$

Then

$$Adj(A) = C\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \Longrightarrow C = \begin{pmatrix} cos\varphi \sin^2\theta & -cos\varphi cos\theta sin\theta & -sin\varphi \\ -sin\varphi \sin^2\theta & sin\varphi cos\theta sin\theta & -cos\varphi \\ cos\theta sin\theta & \sin^2\theta & 0 \end{pmatrix}\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\Longrightarrow Adj(A) = \begin{pmatrix} cos\varphi \sin^2\theta & cos\varphi cos\theta sin\theta & -sin\varphi \\ sin\varphi \sin^2\theta & sin\varphi cos\theta sin\theta & cos\varphi \\ cos\theta sin\theta & -\sin^2\theta & 0 \end{pmatrix}$$

Finally we need to find the determinant of our original matrix A given by

$$A = \begin{pmatrix} cos\varphi sin\theta & sin\varphi sin\theta & cos\theta \\ cos\varphi cos\theta & sin\varphi cos\theta & -sin\theta \\ -sin\varphi sin\theta & cos\varphi sin\theta & 0 \end{pmatrix}$$

$$\Longrightarrow \det(A) = cos\varphi sin\theta\begin{pmatrix} sin\varphi cos\theta & -sin\theta \\ cos\varphi sin\theta & 0 \end{pmatrix} - sin\varphi sin\theta\begin{pmatrix} cos\varphi cos\theta & -sin\theta \\ -sin\varphi sin\theta & 0 \end{pmatrix}$$
$$+ cos\theta\begin{pmatrix} cos\varphi cos\theta & sin\varphi cos\theta \\ -sin\varphi sin\theta & cos\varphi sin\theta \end{pmatrix}$$

$$= cos\varphi sin\theta(0 + cos\varphi \sin^2\theta) - sin\varphi sin\theta(0 - sin\varphi \sin^2\theta) + cos\theta(\cos^2\varphi\, cos\theta sin\theta$$
$$+ \sin^2\varphi\, cos\theta sin\theta)$$

$$= \cos^2\varphi\, sin\theta(\sin^2\theta) + \sin^2\varphi\, sin\theta(\sin^2\theta) + \cos^2\varphi \cos^2\theta\, sin\theta + \sin^2\varphi \cos^2\theta\, sin\theta$$

$$= sin\theta(\sin^2\theta)(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta\, sin\theta(\cos^2\varphi + \sin^2\varphi)$$

$$= sin\theta(\sin^2\theta) + \cos^2\theta\, sin\theta = sin\theta$$

$$\Longrightarrow \det(A) = sin\theta$$

Finally

$$A^{-1} = \frac{Adj(A)}{\det(A)} = \frac{1}{sin\theta}\begin{pmatrix} cos\varphi \sin^2\theta & cos\varphi cos\theta sin\theta & -sin\varphi \\ sin\varphi \sin^2\theta & sin\varphi cos\theta sin\theta & cos\varphi \\ cos\theta sin\theta & -\sin^2\theta & 0 \end{pmatrix}$$

Therefore

$$A^{-1} = \frac{1}{sin\theta}\begin{pmatrix} cos\varphi \sin^2\theta & cos\varphi cos\theta sin\theta & -sin\varphi \\ sin\varphi \sin^2\theta & sin\varphi cos\theta sin\theta & cos\varphi \\ cos\theta sin\theta & -\sin^2\theta & 0 \end{pmatrix}$$

To ascertain the correctness of the inverse matrix $\boldsymbol{A^{-1}}$, let

$$\boldsymbol{B = A^{-1}A = A^{-1}}\begin{pmatrix} \boldsymbol{cos\varphi sin\theta} & \boldsymbol{sin\varphi sin\theta} & \boldsymbol{cos\theta} \\ \boldsymbol{cos\varphi cos\theta} & \boldsymbol{sin\varphi cos\theta} & \boldsymbol{-sin\theta} \\ \boldsymbol{-sin\varphi sin\theta} & \boldsymbol{cos\varphi sin\theta} & \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{B_{11}} & \boldsymbol{B_{12}} & \boldsymbol{B_{13}} \\ \boldsymbol{B_{21}} & \boldsymbol{B_{22}} & \boldsymbol{B_{23}} \\ \boldsymbol{B_{31}} & \boldsymbol{B_{32}} & \boldsymbol{B_{33}} \end{pmatrix}$$

$$\boldsymbol{B=\frac{1}{sin\theta}\begin{pmatrix} cos\varphi\, sin^2\theta & cos\varphi cos\theta sin\theta & -sin\varphi \\ sin\varphi\, sin^2\theta & sin\varphi cos\theta sin\theta & cos\varphi \\ cos\theta sin\theta & -sin^2\theta & 0 \end{pmatrix}\begin{pmatrix} cos\varphi sin\theta & sin\varphi sin\theta & cos\theta \\ cos\varphi cos\theta & sin\varphi cos\theta & -sin\theta \\ -sin\varphi sin\theta & cos\varphi sin\theta & 0 \end{pmatrix}}$$

$$\boldsymbol{\Rightarrow B_{11}=\frac{1}{sin\theta}(cos\varphi\, sin^2\theta \quad cos\varphi cos\theta sin\theta \quad -sin\varphi)\begin{pmatrix} cos\varphi sin\theta \\ cos\varphi cos\theta \\ -sin\varphi sin\theta \end{pmatrix}}$$

$$\boldsymbol{B_{11}=\frac{1}{sin\theta}(cos^2\varphi\, sin^2\theta\, sin\theta+cos^2\varphi\, cos^2\theta\, sin\theta+sin^2\varphi\, sin\theta)}$$

$$\boldsymbol{=\frac{1}{sin\theta}[cos^2\varphi sin\theta\,(sin^2\theta+cos^2\theta)+sin^2\varphi\, sin\theta)]}$$

$$\boldsymbol{=\frac{1}{sin\theta}(cos^2\varphi sin\theta+sin^2\varphi\, sin\theta)=\frac{sin\theta}{sin\theta}=1}$$

$$\boldsymbol{\Rightarrow B_{11}=1}$$

Similar procedures apply in getting the rest of the values of the matrix; that is:

$$\boldsymbol{B_{12}=\frac{1}{sin\theta}(cos\varphi\, sin^2\theta \quad cos\varphi cos\theta sin\theta \quad -sin\varphi)\begin{pmatrix} sin\varphi sin\theta \\ sin\varphi cos\theta \\ cos\varphi sin\theta \end{pmatrix}\Rightarrow B_{12}}$$

$$\boldsymbol{=\frac{1}{sin\theta}(cos\varphi sin\varphi\, sin^2\theta\, sin\theta+cos\varphi\, sin\varphi cos^2\theta\, sin\theta}$$

$$\boldsymbol{-cos\varphi sin\varphi sin\theta)}$$

$$\boldsymbol{=\frac{1}{sin\theta}(cos\varphi sin\varphi sin\theta[(sin^2\theta+cos^2\theta)-cos\varphi sin\varphi sin\theta])}$$

$$\boldsymbol{=\frac{1}{sin\theta}(cos\varphi sin\varphi sin\theta-cos\varphi sin\varphi sin\theta)=0}$$

$$\boldsymbol{\Rightarrow B_{12}=0}$$

$$\boldsymbol{B_{13}=\frac{1}{sin\theta}(cos\varphi\, sin^2\theta \quad cos\varphi cos\theta sin\theta \quad -sin\varphi)\begin{pmatrix} cos\theta \\ -sin\theta \\ 0 \end{pmatrix}\Rightarrow B_{13}}$$

$$\boldsymbol{=\frac{1}{sin\theta}(cos\varphi\, sin^2\theta\, cos\theta-cos\varphi cos\theta\, sin^2\theta+0)=\frac{1}{sin\theta}(0)=0}$$

$$\boldsymbol{\Rightarrow B_{13}=0}$$

$$B_{21} = \frac{1}{sin\theta}\,(sin\varphi\, sin^2\,\theta \quad sin\varphi cos\theta sin\theta \quad cos\varphi)\begin{pmatrix} cos\varphi sin\theta \\ cos\varphi cos\theta \\ -sin\varphi sin\theta \end{pmatrix} \Longrightarrow B_{21}$$

$$= \frac{1}{sin\theta}\,(cos\varphi sin\varphi\, sin^2\,\theta\, sin\theta + cos\varphi sin\varphi\, cos^2\,\theta\, sin\theta - cos\varphi sin\varphi sin\theta$$

$$= \frac{1}{sin\theta}\,(cos\varphi sin\varphi sin\theta(sin^2\,\theta + cos^2\,\theta) - cos\varphi sin\varphi sin\theta)$$

$$= \frac{1}{sin\theta}\,(cos\varphi sin\varphi sin\theta - cos\varphi sin\varphi sin\theta) = 0$$

$$\Longrightarrow B_{21} = 0$$

$$B_{22} = \frac{1}{sin\theta}\,(sin\varphi\, sin^2\,\theta \quad sin\varphi cos\theta sin\theta \quad cos\varphi)\begin{pmatrix} sin\varphi sin\theta \\ sin\varphi cos\theta \\ cos\varphi sin\theta \end{pmatrix}$$

$$\Longrightarrow B_{22} = \frac{1}{sin\theta}\,(sin^2\,\varphi\, sin^2\,\theta\, sin\theta + sin^2\,\varphi\, cos^2\,\theta\, sin\theta + cos^2\,\varphi\, sin\theta$$

$$= \frac{1}{sin\theta}\,[sin^2\,\varphi\, sin\theta(sin^2\,\theta + cos^2\,\theta) + cos^2\,\varphi\, sin\theta]$$

$$= \frac{1}{sin\theta}\,[sin^2\,\varphi\, sin\theta + cos^2\,\varphi\, sin\theta] = \frac{sin\theta}{sin\theta}\,[sin^2\,\varphi + cos^2\,\varphi] = 1$$

$$\Longrightarrow B_{22} = 1$$

$$B_{23} = \frac{1}{sin\theta}\,(sin\varphi\, sin^2\,\theta \quad sin\varphi cos\theta sin\theta \quad cos\varphi)\begin{pmatrix} cos\theta \\ sin\theta \\ 0 \end{pmatrix} \Longrightarrow B_{23}$$

$$= \frac{1}{sin\theta}\,(sin\varphi cos\theta\, sin^2\,\theta + sin\varphi cos\theta\, sin^2\,\theta - 0) = \frac{1}{sin\theta}\,(0) = 0$$

$$\Longrightarrow B_{23} = 0$$

$$B_{31} = \frac{1}{sin\theta}\,(cos\theta sin\theta \quad -sin^2\,\theta \quad 0)\begin{pmatrix} cos\varphi sin\theta \\ cos\varphi cos\theta \\ -sin\varphi sin\theta \end{pmatrix} \Longrightarrow B_{31}$$

$$= \beta(cos\varphi cos\theta\, sin^2\,\theta - os\varphi cos\theta\, sin^2\,\theta - 0) = \beta(0) = 0$$

$$\Longrightarrow B_{31} = 0$$

$$\boldsymbol{B_{32} = \frac{1}{sin\theta}(cos\theta sin\theta \quad -sin^2\theta \quad 0)\begin{pmatrix} sin\varphi sin\theta \\ sin\varphi cos\theta \\ cos\varphi sin\theta \end{pmatrix} \Rightarrow B_{32}}$$

$$\boldsymbol{= \frac{1}{sin\theta}(sin\varphi cos\theta\, sin^2\theta - sin\varphi cos\theta\, sin^2\theta + 0) = \frac{1}{sin\theta}(0) = 0}$$

$$\boldsymbol{\Rightarrow B_{32} = 0}$$

$$\boldsymbol{B_{33} = \frac{1}{sin\theta}(cos\theta sin\theta \quad -sin^2\theta \quad 0)\begin{pmatrix} cos\theta \\ -sin\theta \\ 0 \end{pmatrix} \Rightarrow B_{33}}$$

$$\boldsymbol{= \frac{1}{sin\theta}(cos^2\theta\, sin\theta + sin^2\theta\, sin\theta + 0) = \frac{1}{sin\theta}sin\theta(sin^2\theta + cos^2\theta)}$$

$$\boldsymbol{= \frac{sin\theta}{sin\theta} = 1}$$

$$\boldsymbol{\Rightarrow B_{33} = 1}$$

Eventually the above detailed mathematical procedures confirm the fact that

$$\boldsymbol{A^{-1}A = B = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Therefore, the above procedures confirm that A^{-1} and A are inverse matrices to each other. However, this confirmatory test is not part of our problem.

Eventually, we continue as follows:

$$A = \begin{pmatrix} cos\varphi sin\theta & sin\varphi sin\theta & cos\theta \\ cos\varphi cos\theta & sin\varphi cos\theta & -sin\theta \\ -sin\varphi sin\theta & cos\varphi sin\theta & 0 \end{pmatrix};$$

$$\Rightarrow A^{-1} = \begin{pmatrix} cos\varphi sin\theta & cos\varphi cos\theta & -\frac{sin\varphi}{sin\theta} \\ sin\varphi sin\theta & sin\varphi cos\theta & \frac{cos\varphi}{sin\theta} \\ cos\theta & -sin\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}=\begin{pmatrix}cos\varphi sin\theta & cos\varphi cos\theta & -\frac{sin\varphi}{sin\theta}\\ sin\varphi sin\theta & sin\varphi cos\theta & \frac{cos\varphi}{sin\theta}\\ cos\theta & -sin\theta & 0\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{1}{r}\frac{\partial}{\partial\theta}\\ \frac{1}{r}\frac{\partial}{\partial\varphi}\end{pmatrix}$$

$$\Longrightarrow \frac{\partial}{\partial x}=cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}$$

$$\frac{\partial}{\partial y}=sin\varphi sin\theta\frac{\partial}{\partial r}+sin\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}$$

and finally;

$$\frac{\partial}{\partial z}=cos\theta\frac{\partial}{\partial r}-\frac{sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial x}\left(cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$=cos\varphi sin\theta\frac{\partial}{\partial x}\left(\frac{\partial}{\partial r}\right)$$

$$\frac{\partial}{\partial x}=cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}$$

$$\Longrightarrow\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial x}\left(cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\Longrightarrow\frac{\partial^2}{\partial x^2}=cos\varphi sin\theta\frac{\partial}{\partial r}\left[cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$+\ cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left[cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\left[cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]=\alpha+\beta+\gamma$$

where;

$$\alpha=cos\varphi sin\theta\frac{\partial}{\partial r}\left[cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\beta=\ cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left[cos\varphi sin\theta\frac{\partial}{\partial r}+cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

and

$$\gamma = -\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\left[cos\varphi sin\theta\frac{\partial}{\partial r} + cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

$$\alpha = cos\varphi sin\theta\frac{\partial}{\partial r}\left[cos\varphi sin\theta\frac{\partial}{\partial r} + cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\alpha = cos\varphi sin\theta\left[os\varphi sin\theta\frac{\partial^2}{\partial r^2} + cos\varphi cos\theta\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\theta}\right)\right)\right.$$

$$\left. - \frac{sin\varphi}{sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial\varphi} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\varphi}\right)\right)\right]$$

$$\alpha = cos\varphi sin\theta\left[os\varphi sin\theta\frac{\partial^2}{\partial r^2} + cos\varphi cos\theta\left(\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)\right.$$

$$\left. - \frac{sin\varphi}{sin\theta}\left(\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial\varphi} + \frac{1}{r}\frac{\partial}{\partial r\partial\varphi}\right)\right]$$

$$\alpha = cos^2\varphi\sin^2\theta\frac{\partial^2}{\partial r^2} + cos^2\varphi os\theta sin\theta\left(\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)$$

$$- cos\varphi sin\varphi\left(\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial\varphi} + \frac{1}{r}\frac{\partial}{\partial r\partial\varphi}\right)$$

$$\alpha = cos^2\varphi\sin^2\theta\frac{\partial^2}{\partial r^2} - \left(\frac{os^2\varphi cos\theta sin\theta}{r^2}\right)\frac{\partial}{\partial\theta} + \left(\frac{os^2\varphi cos\theta sin\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta} + \left(\frac{cos\varphi sin\varphi}{r^2}\right)\frac{\partial}{\partial\varphi}$$

$$- \left(\frac{cos\varphi sin\varphi}{r}\right)\frac{\partial}{\partial r\partial\varphi}$$

$$\Longrightarrow \alpha = cos^2\varphi\sin^2\theta\frac{\partial^2}{\partial r^2} + \left(\frac{cos^2\varphi cos\theta sin\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta} - \left(\frac{cos\varphi sin\varphi}{r}\right)\frac{\partial}{\partial r\partial\varphi}$$

$$- \left(\frac{cos^2\varphi cos\theta sin\theta}{r^2}\right)\frac{\partial}{\partial\theta} + \left(\frac{cos\varphi sin\varphi}{r^2}\right)\frac{\partial}{\partial\varphi}$$

$$\beta = cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left[cos\varphi sin\theta\frac{\partial}{\partial r} + cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\beta = \frac{cos\varphi cos\theta}{r}\left[cos\varphi\left(\frac{\partial}{\partial\theta}(sin\theta)\frac{\partial}{\partial r} + sin\theta\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\right)\right)\right.$$
$$+\left(\frac{cos\varphi}{r}\right)\left(\frac{\partial}{\partial\theta}(cos\theta)\frac{\partial}{\partial\theta} + cos\theta\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right)\right)$$
$$\left.-\left(\frac{sin\varphi}{r}\right)\left(\frac{\partial}{\partial\theta}\left(\frac{1}{sin\theta}\right)\frac{\partial}{\partial\varphi} + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\varphi}\right)\right)\right]$$

$$\beta = \frac{cos\varphi cos\theta}{r}\left[cos\varphi\left(cos\theta\frac{\partial}{\partial r} + sin\theta\frac{\partial^2}{\partial\theta\partial r}\right) + \left(\frac{cos\varphi}{r}\right)\left(-sin\theta\frac{\partial}{\partial\theta} + cos\theta\frac{\partial^2}{\partial\theta^2}\right)\right.$$
$$\left.-\left(\frac{sin\varphi}{r}\right)\left(\left(-\frac{cos\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\varphi} + \frac{1}{sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right)\right]$$

$$\beta = \frac{cos\varphi cos\theta}{r}\left[\left(cos\varphi cos\theta\frac{\partial}{\partial r} + cos\varphi sin\theta\frac{\partial^2}{\partial\theta\partial r}\right) + \left(-\frac{sin\theta cos\varphi}{r}\frac{\partial}{\partial\theta} + \frac{cos\theta cos\varphi}{r}\frac{\partial^2}{\partial\theta^2}\right)\right.$$
$$\left.-\left(-\frac{sin\varphi cos\theta}{r\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{sin\varphi}{rsin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right)\right]$$

$$\beta = \left(\frac{\cos^2\varphi\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\varphi\, cos\theta sin\theta}{r}\frac{\partial^2}{\partial\theta\partial r}\right)$$
$$+\left(-\frac{\cos^2\varphi\, cos\theta sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\varphi\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$$
$$+\left(\frac{\cos^2\theta\, cos\varphi sin\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi} - \frac{cos\varphi cos\theta sin\varphi}{r^2 sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right)$$

$$\Longrightarrow \beta = \left(\frac{\cos^2\varphi\cos^2\theta}{r^2}\right)\frac{\partial^2}{\partial\theta^2} + \left(\frac{\cos^2\varphi\, cos\theta sin\theta}{r}\right)\frac{\partial^2}{\partial\theta\partial r} - \left(\frac{cos\varphi cos\theta sin\varphi}{r^2 sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\varphi}$$
$$+\left(\frac{\cos^2\varphi\cos^2\theta}{r}\right)\frac{\partial}{\partial r} - \left(\frac{\cos^2\varphi\, cos\theta sin\theta}{r^2}\right)\frac{\partial}{\partial\theta} + \left(\frac{\cos^2\theta\, cos\varphi sin\varphi}{r^2\sin^2\theta}\right)\frac{\partial}{\partial\varphi}$$

$$\gamma = -\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\left[cos\varphi sin\theta\frac{\partial}{\partial r} + cos\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\gamma = -\frac{sin\varphi}{rsin\theta}\left[sin\theta\left(\frac{\partial}{\partial\varphi}(cos\varphi)\frac{\partial}{\partial r}+cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}\right)\right)\right.$$
$$+\frac{cos\theta}{r}\left(\frac{\partial}{\partial\varphi}(cos\varphi)\frac{\partial}{\partial\theta}+cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\theta}\right)\right)$$
$$\left.-\frac{1}{rsin\theta}\left(\frac{\partial}{\partial\varphi}(sin\varphi)\frac{\partial}{\partial\varphi}+sin\varphi\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\varphi}\right)\right)\right]$$

$$\gamma = -\frac{sin\varphi}{rsin\theta}\left[sin\theta\left(-sin\varphi\frac{\partial}{\partial r}+cos\varphi\frac{\partial^2}{\partial\varphi\partial r}\right)+\frac{cos\theta}{r}\left(-sin\varphi\frac{\partial}{\partial\theta}+cos\varphi\frac{\partial^2}{\partial\varphi\partial\theta}\right)\right.$$
$$\left.-\frac{1}{rsin\theta}\left(cos\varphi\frac{\partial}{\partial\varphi}+sin\varphi\frac{\partial^2}{\partial\varphi^2}\right)\right]$$

$$\gamma = -\frac{sin\varphi}{rsin\theta}\left[\left(-sin\theta sin\varphi\frac{\partial}{\partial r}+sin\theta cos\varphi\frac{\partial^2}{\partial r\partial\varphi}\right)+\left(-\frac{cos\theta sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{cos\theta cos\varphi}{r}\frac{\partial^2}{\partial\theta\partial\varphi}\right)\right.$$
$$\left.-\left(\frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}+\frac{sin\varphi}{rsin\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right]$$

$$\gamma = \left[\left(\frac{\sin^2\varphi}{r}\frac{\partial}{\partial r}-\frac{sin\varphi cos\varphi}{r}\frac{\partial^2}{\partial r\partial\varphi}\right)+\left(\frac{\sin^2\varphi\, cos\theta}{r^2sin\theta}\frac{\partial}{\partial\theta}-\frac{cos\theta cos\varphi sin\varphi}{r^2sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right)\right.$$
$$\left.+\left(\frac{sin\varphi cos\varphi}{r^2sin^2\theta}\frac{\partial}{\partial\varphi}+\frac{sin^2\varphi}{r^2sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\right]$$

$$\gamma = \frac{\sin^2\varphi}{r}\frac{\partial}{\partial r}-\frac{sin\varphi cos\varphi}{r}\frac{\partial^2}{\partial r\partial\varphi}+\frac{\sin^2\varphi\, cos\theta}{r^2sin\theta}\frac{\partial}{\partial\theta}-\frac{cos\theta cos\varphi sin\varphi}{r^2sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}+\frac{sin\varphi cos\varphi}{r^2sin^2\theta}\frac{\partial}{\partial\varphi}$$
$$+\frac{sin^2\varphi}{r^2sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Longrightarrow \gamma = \left(\frac{sin^2\varphi}{r^2sin^2\theta}\right)\frac{\partial^2}{\partial\varphi^2}-\left(\frac{sin\varphi cos\varphi}{r}\right)\frac{\partial^2}{\partial r\partial\varphi}-\left(\frac{cos\theta cos\varphi sin\varphi}{r^2sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\varphi}+\left(\frac{\sin^2\varphi}{r}\right)\frac{\partial}{\partial r}$$
$$+\left(\frac{sin^2\varphi\, cos\theta}{r^2sin\theta}\right)\frac{\partial}{\partial\theta}+\left(\frac{sin\varphi cos\varphi}{r^2sin^2\theta}\right)\frac{\partial}{\partial\varphi}$$

$$\Rightarrow \alpha + \beta + \gamma = \frac{\partial^2}{\partial x^2}$$

$$= cos^2\varphi \sin^2\theta \frac{\partial^2}{\partial r^2} + \left(\frac{cos^2\varphi cos\theta sin\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta} - \left(\frac{cos\varphi sin\varphi}{r}\right)\frac{\partial}{\partial r\partial\varphi}$$

$$- \left(\frac{cos^2\varphi cos\theta sin\theta}{r^2}\right)\frac{\partial}{\partial\theta} + \left(\frac{cos\varphi sin\varphi}{r^2}\right)\frac{\partial}{\partial\varphi} + \left(\frac{cos^2\varphi\, cos^2\theta}{r^2}\right)\frac{\partial^2}{\partial\theta^2}$$

$$+ \left(\frac{cos^2\varphi\, cos\theta sin\theta}{r}\right)\frac{\partial^2}{\partial\theta\partial r} - \left(\frac{cos\varphi cos\theta sin\varphi}{r^2 sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\varphi} + \left(\frac{cos^2\varphi\, cos^2\theta}{r}\right)\frac{\partial}{\partial r}$$

$$- \left(\frac{cos^2\varphi\, cos\theta sin\theta}{r^2}\right)\frac{\partial}{\partial\theta} + \left(\frac{cos\varphi\, cos^2\theta\, sin\varphi}{r^2\, sin^2\theta}\right)\frac{\partial}{\partial\varphi} + \left(\frac{sin^2\varphi}{r^2 sin^2\theta}\right)\frac{\partial^2}{\partial\varphi^2}$$

$$- \left(\frac{sin\varphi cos\varphi}{r}\right)\frac{\partial^2}{\partial r\partial\varphi} - \left(\frac{cos\theta cos\varphi sin\varphi}{r^2 sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\varphi} + \left(\frac{\sin^2\varphi}{r}\right)\frac{\partial}{\partial r}$$

$$+ \left(\frac{sin^2\varphi\, cos\theta}{r^2 sin\theta}\right)\frac{\partial}{\partial\theta} + \left(\frac{sin\varphi cos\varphi}{r^2 sin^2\theta}\right)\frac{\partial}{\partial\varphi}$$

$\frac{\partial^2}{\partial x^2}$ – *COEFFICIENTS OF DERIVATIVES SUMMARY TABLE* – 1

DERIVATIVE	COEFFICIENT(S)		
	α	β	γ
$\frac{\partial^2}{\partial r^2}$	$cos^2\varphi \sin^2\theta$	0	0
$\frac{\partial^2}{\partial\theta^2}$	0	$\frac{\cos^2\theta\, cos^2\varphi}{r^2}$	0
$\frac{\partial^2}{\partial\varphi^2}$	0	0	$\frac{sin^2\varphi}{r^2 sin^2\theta}$
$\frac{\partial^2}{\partial r\partial\theta}$	$\frac{os^2\varphi cos\theta sin\theta}{r}$	$\frac{cos^2\varphi cos\theta sin\theta}{r}$	0
$\frac{\partial^2}{\partial r\partial\varphi}$	$-\frac{cos\varphi sin\varphi}{r}$	0	$-\frac{sin\varphi cos\varphi}{r}$
$\frac{\partial^2}{\partial\theta\partial\varphi}$	0	$-\frac{cos\varphi cos\theta sin\varphi}{r^2 sin\theta}$	$-\frac{cos\theta cos\varphi sin\varphi}{r^2 sin\theta}$
$\frac{\partial}{\partial r}$	0	$\frac{cos^2\theta\, cos^2\varphi}{r}$	$\frac{\sin^2\varphi}{r}$
$\frac{\partial}{\partial\theta}$	$-\frac{cos^2\varphi cos\theta sin\theta}{r^2}$	$-\frac{cos\theta sin\theta cos^2\varphi}{r^2}$	$\frac{sin^2\varphi\, cos\theta}{r^2 sin\theta}$
$\frac{\partial}{\partial\varphi}$	$\frac{cos\varphi sin\varphi}{r^2}$	$\frac{cos\varphi\, cos^2\theta\, sin\varphi}{r^2\, sin^2\theta}$	$\frac{sin\varphi cos\varphi}{r^2 sin^2\theta}$

Table 1a

$$\frac{\partial^2}{\partial x^2} - \textbf{COEFFICIENTS OF DERIVATIVES SUMMARY TABLE} - \mathbf{2}$$

DERIVATIVE	COEFFICIENT
$\frac{\partial^2}{\partial r^2}$	$cos^2\varphi \sin^2\theta$
$\frac{\partial^2}{\partial \theta^2}$	$\frac{\cos^2\theta\, cos^2\varphi}{r^2}$
$\frac{\partial^2}{\partial \varphi^2}$	$\frac{sin^2\varphi}{r^2 sin^2\theta}$
$\frac{\partial^2}{\partial r \partial \theta}$	$\frac{2cos\theta sin\theta cos^2\varphi}{r}$
$\frac{\partial^2}{\partial r \partial \varphi}$	$-\frac{2sin\varphi cos\varphi}{r}$
$\frac{\partial^2}{\partial \theta \partial \varphi}$	$-\frac{2cos\theta cos\varphi sin\varphi}{r^2 sin\theta}$
$\frac{\partial}{\partial r}$	$\frac{cos^2\theta\, cos^2\varphi}{r} + \frac{\sin^2\varphi}{r}$
$\frac{\partial}{\partial \theta}$	$-\frac{2cos\theta sin\theta cos^2\varphi}{r^2} + \frac{sin^2\varphi\, cos\theta}{r^2 sin\theta}$
$\frac{\partial}{\partial \varphi}$	$\frac{cos\varphi sin\varphi}{r^2} + \frac{cos\varphi\, cos^2\theta\, sin\varphi}{r^2\, sin^2\theta} + \frac{sin\varphi cos\varphi}{r^2 sin^2\theta}$

Table 1b

Similarly:

$$\frac{\partial}{\partial y} = sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{r sin\theta}\frac{\partial}{\partial \varphi}$$

$$\Longrightarrow \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{r sin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$\Longrightarrow \frac{\partial^2}{\partial y^2} = sin\varphi sin\theta \frac{\partial}{\partial r}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{r sin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$+ sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{r sin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$+ \frac{cos\varphi}{r sin\theta}\frac{\partial}{\partial \varphi}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{r sin\theta}\frac{\partial}{\partial \varphi}\right] = \alpha' + \beta' + \gamma'$$

where

$$\alpha' = sin\varphi sin\theta \frac{\partial}{\partial r}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$\beta' = sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial \varphi}\right]$$

and

$$\gamma' = \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial \varphi}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial \varphi}\right]$$

Now beginning working on α'we have

$$\alpha' = sin\varphi sin\theta \frac{\partial}{\partial r}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$\Longrightarrow \alpha' = sin\varphi sin\theta \left[sin\varphi sin\theta \frac{\partial^2}{\partial r^2} + sin\varphi cos\theta \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\right)\right)\right.$$

$$\left. + \frac{cos\varphi}{sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial \varphi} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial \varphi}\right)\right)\right]$$

$$\alpha' = sin\varphi sin\theta \left[sin\varphi sin\theta \frac{\partial^2}{\partial r^2} + sin\varphi cos\theta \left(-\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta}\right) + \frac{cos\varphi}{sin\theta}\left(-\frac{1}{r^2}\frac{\partial}{\partial \varphi} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \varphi}\right)\right]$$

$$\alpha' = sin\varphi sin\theta \left[sin\varphi sin\theta \frac{\partial^2}{\partial r^2} - \frac{sin\varphi cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{sin\varphi cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{cos\varphi}{r^2 sin\theta}\frac{\partial}{\partial \varphi} + \frac{cos\varphi}{rsin\theta}\frac{\partial^2}{\partial r \partial \varphi}\right]$$

$$\alpha' = \sin^2\varphi \sin^2\theta \frac{\partial^2}{\partial r^2} - \frac{\sin^2\varphi sin\theta\, cos\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin^2\varphi sin\theta\, cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{sin\varphi cos\varphi}{r^2}\frac{\partial}{\partial \varphi}$$

$$+ \frac{sin\varphi cos\varphi}{r}\frac{\partial^2}{\partial r \partial \varphi}$$

$$\Longrightarrow \alpha' = (\sin^2\varphi \sin^2\theta)\frac{\partial^2}{\partial r^2} + \left(\frac{\sin^2\varphi sin\theta\, cos\theta}{r}\right)\frac{\partial^2}{\partial r \partial \theta} + \left(\frac{sin\varphi cos\varphi}{r}\right)\frac{\partial^2}{\partial r \partial \varphi}$$

$$- \left(\frac{\sin^2\varphi sin\theta\, cos\theta}{r^2}\right)\frac{\partial}{\partial \theta} - \left(\frac{sin\varphi cos\varphi}{r^2}\right)\frac{\partial}{\partial \varphi}$$

Also;

$$\beta' = sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta}\left[sin\varphi sin\theta \frac{\partial}{\partial r} + sin\varphi cos\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial \varphi}\right]$$

$$\Rightarrow \beta' = \frac{sin\varphi cos\theta}{r}\left[sin\varphi\left(\frac{\partial}{\partial\theta}(sin\theta)\frac{\partial}{\partial r} + sin\theta\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\right)\right)\right.$$
$$+\frac{sin\varphi}{r}\left(\frac{\partial}{\partial\theta}(cos\theta)\frac{\partial}{\partial\theta} + cos\theta\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\right)\right)$$
$$\left.+\frac{cos\varphi}{r}\left(\frac{\partial}{\partial\theta}\left(\frac{1}{sin\theta}\right)\frac{\partial}{\partial\varphi} + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\varphi}\right)\right)\right]$$
$$= \frac{sin\varphi cos\theta}{r}\left[sin\varphi\left(cos\theta\frac{\partial}{\partial r} + sin\theta\frac{\partial^2}{\partial\theta\partial r}\right) + \frac{sin\varphi}{r}\left(-sin\theta\frac{\partial}{\partial\theta} + cos\theta\frac{\partial^2}{\partial\theta^2}\right)\right.$$
$$\left.+\frac{cos\varphi}{r}\left(-\frac{cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{1}{sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right)\right]$$
$$= \frac{sin\varphi cos\theta}{r}\left[\left(cos\theta sin\varphi\frac{\partial}{\partial r} + sin\theta sin\varphi\frac{\partial^2}{\partial\theta\partial r}\right) + \left(-\frac{sin\theta sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{cos\theta sin\varphi}{r}\frac{\partial^2}{\partial\theta^2}\right)\right.$$
$$\left.+\left(-\frac{cos\theta cos\varphi}{r\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{cos\varphi}{rsin\theta}\frac{\partial^2}{\partial\theta\partial\varphi}\right)\right]$$
$$\beta' = \frac{\sin^2\varphi\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\varphi\, cos\theta sin\theta}{r}\frac{\partial^2}{\partial\theta\partial r} - \frac{\sin^2\varphi\, cos\theta sin\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\varphi\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$
$$-\frac{\cos^2\theta\, sin\varphi cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi} + \frac{sin\varphi cos\varphi cos\theta}{r^2 sin\,\theta}\frac{\partial^2}{\partial\theta\partial\varphi}$$

$$\Rightarrow \beta' = \left(\frac{\sin^2\varphi\cos^2\theta}{r^2}\right)\frac{\partial^2}{\partial\theta^2} + \left(\frac{\sin^2\varphi\, cos\theta sin\theta}{r}\right)\frac{\partial^2}{\partial\theta\partial r} + \left(\frac{sin\varphi cos\varphi cos\theta}{r^2 sin\,\theta}\right)\frac{\partial^2}{\partial\theta\partial\varphi}$$
$$+\left(\frac{\sin^2\varphi\cos^2\theta}{r}\right)\frac{\partial}{\partial r} - \left(\frac{\sin^2\varphi\, cos\theta sin\theta}{r^2}\right)\frac{\partial}{\partial\theta} - \left(\frac{\cos^2\theta\, sin\varphi cos\varphi}{r^2\sin^2\theta}\right)\frac{\partial}{\partial\varphi}$$

Likewise

$$\gamma' = \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\left[sin\varphi sin\theta\frac{\partial}{\partial r} + sin\varphi cos\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\Longrightarrow \gamma' = \frac{cos\varphi}{rsin\theta}\left[sin\theta\left(\frac{\partial}{\partial\varphi}(sin\varphi)\frac{\partial}{\partial r} + sin\varphi\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial r}\right)\right)\right.$$
$$+\frac{cos\theta}{r}\left(\frac{\partial}{\partial\varphi}(sin\varphi)\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\theta}\right)\right)$$
$$\left.+\frac{1}{rsin\theta}\left(\frac{\partial}{\partial\varphi}(cos\varphi)\frac{\partial}{\partial\varphi} + cos\varphi\frac{\partial^2}{\partial\varphi^2}\right)\right]$$

$$\gamma' = \frac{cos\varphi}{rsin\theta}\left[sin\theta\left(cos\varphi\frac{\partial}{\partial r} + sin\varphi\frac{\partial^2}{\partial\varphi\partial r}\right) + \frac{cos\theta}{r}\left(cos\varphi\frac{\partial}{\partial\theta} + sin\varphi\frac{\partial^2}{\partial\varphi\partial\theta}\right)\right.$$
$$\left.+\frac{1}{rsin\theta}\left(cos\varphi\frac{\partial^2}{\partial\varphi^2} - sin\varphi\frac{\partial}{\partial\varphi}\right)\right]$$

$$\gamma' = \frac{cos\varphi}{rsin\theta}\left[sin\theta cos\varphi\frac{\partial}{\partial r} + sin\theta sin\varphi\frac{\partial^2}{\partial\varphi\partial r} + \frac{cos\varphi cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{sin\varphi cos\theta}{r}\frac{\partial^2}{\partial\varphi\partial\theta} + \frac{cos\varphi}{rsin\theta}\frac{\partial^2}{\partial\varphi^2}\right.$$
$$\left.-\frac{sin\varphi}{rsin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\gamma' = \frac{\cos^2\varphi}{r}\frac{\partial}{\partial r} + \frac{sin\varphi cos\varphi}{r}\frac{\partial^2}{\partial\varphi\partial r} + \frac{\cos^2\varphi\, cos\theta}{r^2 sin\theta}\frac{\partial}{\partial\theta} + \frac{sin\varphi cos\varphi cos\theta}{r^2 sin\theta}\frac{\partial^2}{\partial\varphi\partial\theta} + \frac{\cos^2\varphi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$
$$-\frac{cos\varphi sin\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}$$

$$\Longrightarrow \gamma' = \left(\frac{\cos^2\varphi}{r^2\sin^2\theta}\right)\frac{\partial^2}{\partial\varphi^2} + \left(\frac{sin\varphi cos\varphi}{r}\right)\frac{\partial^2}{\partial\varphi\partial r} + \left(\frac{sin\varphi cos\varphi cos\theta}{r^2 sin\theta}\right)\frac{\partial^2}{\partial\varphi\partial\theta} + \left(\frac{\cos^2\varphi}{r}\right)\frac{\partial}{\partial r}$$
$$+\left(\frac{\cos^2\varphi\, cos\theta}{r^2 sin\theta}\right)\frac{\partial}{\partial\theta} - \left(\frac{cos\varphi sin\varphi}{r^2\sin^2\theta}\right)\frac{\partial}{\partial\varphi}$$

Finally;

$$\frac{\partial^2}{\partial y^2} = \alpha' + \beta' + \gamma$$

$$\Longrightarrow \frac{\partial^2}{\partial y^2} = (sin^2\,\varphi\, sin^2\,\theta)\frac{\partial^2}{\partial r^2} + \left(\frac{sin^2\,\varphi sin\theta\, cos\theta}{r}\right)\frac{\partial^2}{\partial r\partial\theta} + \left(\frac{sin\varphi cos\varphi}{r}\right)\frac{\partial^2}{\partial r\partial\varphi}$$
$$- \left(\frac{sin^2\,\varphi sin\theta\, cos\theta}{r^2}\right)\frac{\partial}{\partial\theta} - \left(\frac{sin\varphi cos\varphi}{r^2}\right)\frac{\partial}{\partial\varphi} + \left(\frac{\sin^2\varphi\cos^2\theta}{r^2}\right)\frac{\partial^2}{\partial\theta^2}$$
$$+ \left(\frac{\sin^2\varphi\, cos\theta sin\theta}{r}\right)\frac{\partial^2}{\partial\theta\partial r} + \left(\frac{sin\varphi cos\varphi cos\theta}{r^2 sin\theta}\right)\frac{\partial^2}{\partial\theta\partial\varphi} + \left(\frac{\sin^2\varphi\cos^2\theta}{r}\right)\frac{\partial}{\partial r}$$
$$- \left(\frac{\sin^2\varphi\, cos\theta sin\theta}{r^2}\right)\frac{\partial}{\partial\theta} - \left(\frac{\cos^2\theta\, sin\varphi cos\varphi}{r^2\sin^2\theta}\right)\frac{\partial}{\partial\varphi} + \left(\frac{cos^2\,\varphi}{r^2\, sin^2\,\theta}\right)\frac{\partial^2}{\partial\varphi^2}$$
$$+ \left(\frac{sin\varphi cos\varphi}{r}\right)\frac{\partial^2}{\partial\varphi\partial r} + \left(\frac{sin\varphi cos\varphi cos\theta}{r^2 sin\theta}\right)\frac{\partial^2}{\partial\varphi\partial\theta} + \left(\frac{cos^2\,\varphi}{r}\right)\frac{\partial}{\partial r}$$
$$+ \left(\frac{cos^2\,\varphi\, cos\theta}{r^2 sin\theta}\right)\frac{\partial}{\partial\theta} - \left(\frac{cos\varphi sin\varphi}{r^2\, sin^2\,\theta}\right)\frac{\partial}{\partial\varphi}$$

$\frac{\partial^2}{\partial y^2}$ – *COEFFICIENTS OF DERIVATIVES SUMMARY TABLE* – 1

DERIVATIVE	COEFFICIENT(S)		
	α'	β'	γ'
$\frac{\partial^2}{\partial r^2}$	$\sin^2\varphi\sin^2\theta$	0	0
$\frac{\partial^2}{\partial\theta^2}$	0	$\frac{\sin^2\varphi\cos^2\theta}{r^2}$	0
$\frac{\partial^2}{\partial\varphi^2}$	0	0	$\frac{cos^2\,\varphi}{r^2\, sin^2\,\theta}$
$\frac{\partial^2}{\partial r\partial\theta}$	$\frac{\sin^2\varphi sin\theta\, cos\theta}{r}$	$\frac{sin^2\,\varphi\, cos\theta sin\theta}{r}$	0
$\frac{\partial^2}{\partial r\partial\varphi}$	$\frac{sin\varphi cos\varphi}{r}$	0	$\frac{sin\varphi cos\varphi}{r}$
$\frac{\partial^2}{\partial\theta\partial\varphi}$	0	$\frac{sin\varphi cos\varphi cos\theta}{r^2 sin\theta}$	$\frac{sin\varphi cos\varphi cos\theta}{r^2 sin\theta}$
$\frac{\partial}{\partial r}$	0	$\frac{sin^2\,\varphi\, cos^2\,\theta}{r}$	$\frac{\cos^2\varphi}{r}$
$\frac{\partial}{\partial\theta}$	$-\frac{\sin^2\varphi sin\theta\, cos\theta}{r^2}$	$-\frac{\sin^2\varphi\, cos\theta sin\theta}{r^2}$	$\frac{cos^2\,\varphi\, cos\theta}{r^2 sin\theta}$
$\frac{\partial}{\partial\varphi}$	$-\frac{sin\varphi cos\varphi}{r^2}$	$-\frac{cos^2\,\theta\, sin\varphi cos\varphi}{r^2\, sin^2\,\theta}$	$-\frac{cos\varphi sin\varphi}{r^2\sin^2\theta}$

Table 2a

$\frac{\partial^2}{\partial y^2}$ – ***COEFFICIENTS OF DERIVATIVES SUMMARY TABLE – 2***

DERIVATIVE	COEFFICIENT
$\frac{\partial^2}{\partial r^2}$	$\sin^2\varphi \sin^2\theta$
$\frac{\partial^2}{\partial \theta^2}$	$\frac{\sin^2\varphi \cos^2\theta}{r^2}$
$\frac{\partial^2}{\partial \varphi^2}$	$\frac{\cos^2\varphi}{r^2 \sin^2\theta}$
$\frac{\partial^2}{\partial r \partial \theta}$	$\frac{2\sin^2\varphi sin\theta\, cos\theta}{r}$
$\frac{\partial^2}{\partial r \partial \varphi}$	$\frac{2sin\varphi cos\varphi}{r}$
$\frac{\partial^2}{\partial \theta \partial \varphi}$	$\frac{2sin\varphi cos\varphi cos\theta}{r^2 sin\theta}$
$\frac{\partial}{\partial r}$	$\frac{sin^2\varphi\, cos^2\theta}{r} + \frac{\cos^2\varphi}{r}$
$\frac{\partial}{\partial \theta}$	$-\frac{2\sin^2\varphi sin\theta\, cos\theta}{r^2} + \frac{cos^2\varphi\, cos\theta}{r^2 sin\theta}$
$\frac{\partial}{\partial \varphi}$	$-\left(\frac{sin\varphi cos\varphi}{r^2} + \frac{cos^2\theta\, sin\varphi cos\varphi}{r^2\, sin^2\theta} + \frac{cos\varphi sin\varphi}{r^2 \sin^2\theta}\right)$

Table 2b

Lastly, we have

$$\frac{\partial}{\partial z} = cos\theta \frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial z}\left(cos\theta \frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial z^2} = cos\theta \frac{\partial}{\partial r}\left(cos\theta \frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial \theta}\right) - \frac{sin\theta}{r}\frac{\partial}{\partial \theta}\left(cos\theta \frac{\partial}{\partial r} - \frac{sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= cos\theta \left[cos\theta \frac{\partial^2}{\partial r^2} - \left(sin\theta \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial \theta}\right)\right)\right)\right.$$

$$\left. - \frac{sin\theta}{r}\left(\frac{\partial}{\partial \theta}(cos\theta)\frac{\partial}{\partial r} + cos\theta \frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial r}\right)\right.\right.$$

$$\left.\left. + \frac{sin\theta}{r}\left(\frac{\partial}{\partial \theta}\left(\frac{sin\theta}{r}\right)\frac{\partial}{\partial \theta} + \frac{sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial \theta}\right)\right)\right)\right]$$

$$= cos\theta \left[cos\theta \frac{\partial^2}{\partial r^2} - \left(-\frac{sin\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{sin\theta}{r}\frac{\partial^2}{\partial r \partial \theta}\right) - \frac{sin\theta}{r}\left(-sin\theta \frac{\partial}{\partial r} + cos\theta \frac{\partial^2}{\partial r \partial \theta}\right)\right.$$

$$\left. + \frac{sin\theta}{r}\left(\left(\frac{cos\theta}{r}\right)\frac{\partial}{\partial \theta} + \frac{sin\theta}{r}\frac{\partial^2}{\partial \theta^2}\right)\right]$$

$$= cos\theta \left[\left(cos\theta \frac{\partial^2}{\partial r^2} + \frac{sin\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{sin\theta}{r}\frac{\partial^2}{\partial r \partial \theta}\right) + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{sin\theta cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{sin\theta cos\theta}{r^2}\frac{\partial}{\partial \theta}\right.$$

$$\left. + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}\right]$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{sin\theta cos\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{sin\theta cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} - \frac{sin\theta cos\theta}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{sin\theta cos\theta}{r^2}\frac{\partial}{\partial \theta}$$

$$+ \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$\frac{\partial^2}{\partial z^2}$ – *COEFFICIENTS OF DERIVATIVES SUMMARY TABLE*

DERIVATIVE	$\frac{\partial^2}{\partial r^2}$	$\frac{\partial^2}{\partial \theta^2}$	$\frac{\partial^2}{\partial \varphi^2}$	$\frac{\partial^2}{\partial r \partial \theta}$	$\frac{\partial^2}{\partial r \partial \varphi}$	$\frac{\partial^2}{\partial \theta \partial \varphi}$	$\frac{\partial}{\partial r}$	$\frac{\partial}{\partial \theta}$	$\frac{\partial}{\partial \varphi}$
COEFFICIENT	$\cos^2\theta$	$\frac{\sin^2\theta}{r^2}$	0	$-\frac{2sin\theta cos\theta}{r}$	0	0	$\frac{\sin^2\theta}{r}$	$\frac{2sin\theta cos\theta}{r^2}$	0

Table 3

$$\nabla_s^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \textbf{COEFFICIENTS OF DERIVATIVES SUMMARY TABLE} - \mathbf{1}$$

DERIVATIVE	COEFFICIENT
$\frac{\partial^2}{\partial r^2}$	$os^2\varphi \sin^2\theta + (\sin^2\varphi \sin^2\theta) + \cos^2\theta$
$\frac{\partial^2}{\partial \theta^2}$	$\frac{\cos^2\theta\, cos^2\varphi}{r^2} + \left(\frac{\sin^2\varphi \cos^2\theta}{r^2}\right) + \frac{\sin^2\theta}{r^2}$
$\frac{\partial^2}{\partial \varphi^2}$	$\frac{sin^2\varphi}{r^2 sin^2\theta} + \left(\frac{\cos^2\varphi}{r^2 \sin^2\theta}\right) + 0$
$\frac{\partial^2}{\partial r \partial \theta}$	$\frac{2cos\theta sin\theta cos^2\varphi}{r} + \left(\frac{2\sin^2\varphi sin\theta\, cos\theta}{r}\right) - \frac{2sin\theta cos\theta}{r}$
$\frac{\partial^2}{\partial r \partial \varphi}$	$-\frac{2sin\varphi cos\varphi}{r} + \left(\frac{2sin\varphi cos\varphi}{r}\right) + 0$
$\frac{\partial^2}{\partial \theta \partial \varphi}$	$-\frac{2cos\theta cos\varphi sin\varphi}{r^2 sin\theta} + \left(\frac{2sin\varphi cos\varphi cos\theta}{r^2 sin\theta}\right) + 0$
$\frac{\partial}{\partial r}$	$\frac{cos^2\theta\, cos^2\varphi}{r} + \frac{\sin^2\varphi}{r} + \left(\frac{sin^2\varphi\, cos^2\theta}{r} + \frac{\cos^2\varphi}{r}\right) + \frac{\sin^2\theta}{r}$
$\frac{\partial}{\partial \theta}$	$-\frac{2cos\theta sin\theta cos^2\varphi}{r^2} + \frac{sin^2\varphi\, cos\theta}{r^2 sin\theta} + \left(-\frac{2\sin^2\varphi sin\theta\, cos\theta}{r^2} + \frac{cos^2\varphi\, cos\theta}{r^2 sin\theta}\right) + \frac{2sin\theta cos\theta}{r^2}$
$\frac{\partial}{\partial \varphi}$	$\frac{cos\varphi sin\varphi}{r^2} + \frac{cos\varphi\, cos^2\theta\, sin\varphi}{r^2 sin^2\theta} + \frac{sin\varphi cos\varphi}{r^2 sin^2\theta} - \left(\frac{sin\varphi cos\varphi}{r^2} + \frac{cos^2\theta\, sin\varphi cos\varphi}{r^2 sin^2\theta} + \frac{cos\varphi sin\varphi}{r^2 \sin^2\theta}\right) + 0$

SUMMING UP AND SIMPLIFYING THE ABOVE TRIGONOMETRIC RATIO TERMS

coefficients for

$\frac{\partial^2}{\partial r^2}$	$\cos^2\theta + cos^2\varphi \sin^2\theta + \sin^2\varphi \sin^2\theta = 1$

$$\cos^2\theta + cos^2\varphi \sin^2\theta + \sin^2\varphi \sin^2\theta \Longrightarrow \cos^2\theta + \sin^2\theta\,(cos^2\varphi + \sin^2\varphi)$$
$$= \cos^2\theta + \sin^2\theta = 1$$

$\frac{\partial^2}{\partial \theta^2}$	$\frac{\cos^2\theta\, cos^2\varphi}{r^2} + \left(\frac{\sin^2\varphi \cos^2\theta}{r^2}\right) + \frac{\sin^2\theta}{r^2}$

$$\frac{\cos^2\theta\, cos^2\varphi}{r^2}+\frac{\sin^2\varphi\cos^2\theta}{r^2}+\frac{\sin^2\theta}{r^2}=\frac{\cos^2\theta}{r^2}(cos^2\varphi+\sin^2\varphi)+\frac{\sin^2\theta}{r^2}$$
$$=\frac{1}{r^2}(\cos^2\theta+\sin^2\theta)=\frac{1}{r^2}$$

$\boldsymbol{\frac{\partial^2}{\partial\varphi^2}}$	$\frac{sin^2\varphi}{r^2sin^2\theta}+\left(\frac{\cos^2\varphi}{r^2\sin^2\theta}\right)$

$$\frac{sin^2\varphi}{r^2sin^2\theta}+\frac{\cos^2\varphi}{r^2\sin^2\theta}=\frac{1}{r^2sin^2\theta}(\cos^2\varphi+\sin^2\varphi)=\frac{1}{r^2sin^2\theta}$$

$\boldsymbol{\frac{\partial^2}{\partial r\partial\theta}}$	$\frac{2cos\theta sin\theta cos^2\varphi}{r}+\left(\frac{2\sin^2\varphi sin\theta\, cos\theta}{r}\right)-\frac{2sin\theta cos\theta}{r}$

$$\frac{2cos\theta sin\theta cos^2\varphi}{r}+\frac{2\sin^2\varphi sin\theta\, cos\theta}{r}-\frac{2sin\theta cos\theta}{r}$$
$$=\frac{2sin\theta cos\theta}{r}(\cos^2\varphi+\sin^2\varphi)-\frac{2sin\theta cos\theta}{r}=\frac{2sin\theta cos\theta}{r}-\frac{2sin\theta cos\theta}{r}=0$$

$\boldsymbol{\frac{\partial^2}{\partial r\partial\varphi}}$	$-\frac{2sin\varphi cos\varphi}{r}+\left(\frac{2sin\varphi cos\varphi}{r}\right)$

$$-\frac{2sin\varphi cos\varphi}{r}+\frac{2sin\varphi cos\varphi}{r}=0$$

$\boldsymbol{\frac{\partial^2}{\partial\theta\partial\varphi}}$	$-\frac{2cos\theta cos\varphi sin\varphi}{r^2sin\theta}+\left(\frac{2sin\varphi cos\varphi cos\theta}{r^2sin\theta}\right)$

$$-\frac{2cos\theta cos\varphi sin\varphi}{r^2sin\theta}+\frac{2sin\varphi cos\varphi cos\theta}{r^2sin\theta}=0$$

$\boldsymbol{\frac{\partial}{\partial r}}$	$\frac{cos^2\theta\, cos^2\varphi}{r}+\frac{\sin^2\varphi}{r}+\left(\frac{sin^2\varphi\, cos^2\theta}{r}+\frac{\cos^2\varphi}{r}\right)+\frac{\sin^2\theta}{r}$

$$\frac{cos^2\theta\, cos^2\varphi}{r} + \frac{sin^2\varphi\, cos^2\theta}{r} + \frac{\sin^2\varphi}{r} + \frac{\cos^2\varphi}{r} + \frac{\sin^2\theta}{r}$$

$$= \frac{1}{r}[cos^2\theta\,(cos^2\varphi + sin^2\varphi) + (\sin^2\varphi + \cos^2\varphi) + \sin^2\theta]$$

$$= \frac{1}{r}[1 + cos^2\theta + \sin^2\theta] = \frac{2}{r}$$

$\boldsymbol{\frac{\partial}{\partial\theta}}$	$-\frac{2cos\theta sin\theta cos^2\varphi}{r^2} + \frac{sin^2\varphi\, cos\theta}{r^2 sin\theta} + \left(-\frac{2\sin^2\varphi sin\theta\, cos\theta}{r^2} + \frac{cos^2\varphi\, cos\theta}{r^2 sin\theta}\right) + \frac{2sin\theta cos\theta}{r^2}$

$$-\frac{2cos\theta sin\theta cos^2\varphi}{r^2} - \frac{2\sin^2\varphi sin\theta\, cos\theta}{r^2} + \frac{sin^2\varphi\, cos\theta}{r^2 sin\theta} + \frac{cos^2\varphi\, cos\theta}{r^2 sin\theta} + \frac{2sin\theta cos\theta}{r^2}$$

$$= -\frac{2sin\theta cos\theta}{r^2}(cos^2\varphi + sin^2\varphi) + \frac{cos\theta}{r^2 sin\theta}(sin^2\varphi + cos^2\varphi) + \frac{2sin\theta cos\theta}{r^2}$$

$$= -\frac{2sin\theta cos\theta}{r^2} + \frac{2sin\theta cos\theta}{r^2} + \frac{cos\theta}{r^2 sin\theta} = \frac{cos\theta}{r^2 sin\theta}$$

$\boldsymbol{\frac{\partial}{\partial\varphi}}$	$\frac{cos\varphi sin\varphi}{r^2} + \frac{cos\varphi\, cos^2\theta\, sin\varphi}{r^2 sin^2\theta} + \frac{sin\varphi cos\varphi}{r^2 sin^2\theta} - \left(\frac{sin\varphi cos\varphi}{r^2} + \frac{cos^2\theta\, sin\varphi cos\varphi}{r^2 sin^2\theta} + \frac{cos\varphi sin\varphi}{r^2\sin^2\theta}\right) + 0$

$$\frac{cos\varphi sin\varphi}{r^2} - \frac{sin\varphi cos\varphi}{r^2} + \frac{cos\varphi\, cos^2\theta\, sin\varphi}{r^2 sin^2\theta} - \frac{cos^2\theta\, sin\varphi cos\varphi}{r^2 sin^2\theta} + \frac{sin\varphi cos\varphi}{r^2 sin^2\theta} - \frac{cos\varphi sin\varphi}{r^2\sin^2\theta} = 0$$

$$\boldsymbol{\nabla_s^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - COEFFICIENTS\ OF\ DERIVATIVES\ \underline{FINAL\ TABLE}}$$

DERIVATIVE	$\frac{\partial^2}{\partial r^2}$	$\frac{\partial^2}{\partial\theta^2}$	$\frac{\partial^2}{\partial\varphi^2}$	$\frac{\partial^2}{\partial r\partial\theta}$	$\frac{\partial^2}{\partial r\partial\varphi}$	$\frac{\partial^2}{\partial\theta\partial\varphi}$	$\frac{\partial}{\partial r}$	$\frac{\partial}{\partial\theta}$	$\frac{\partial}{\partial\varphi}$
COEFFICIENT	1	$\frac{1}{r^2}$	$\frac{1}{r^2 sin^2\theta}$	0	0	0	$\frac{2}{r}$	$\frac{cos\theta}{r^2 sin\theta}$	0

$$\nabla_s^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \left(\frac{2}{r}\right)\frac{\partial}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2}{\partial\theta^2} + \left(\frac{cos\theta}{r^2 sin\theta}\right)\frac{\partial}{\partial\theta} + \left(\frac{1}{r^2 sin^2\theta}\right)\frac{\partial^2}{\partial\varphi^2}$$

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right] + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

Hence

$$\boldsymbol{\nabla_s^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right] + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial \varphi^2}}$$

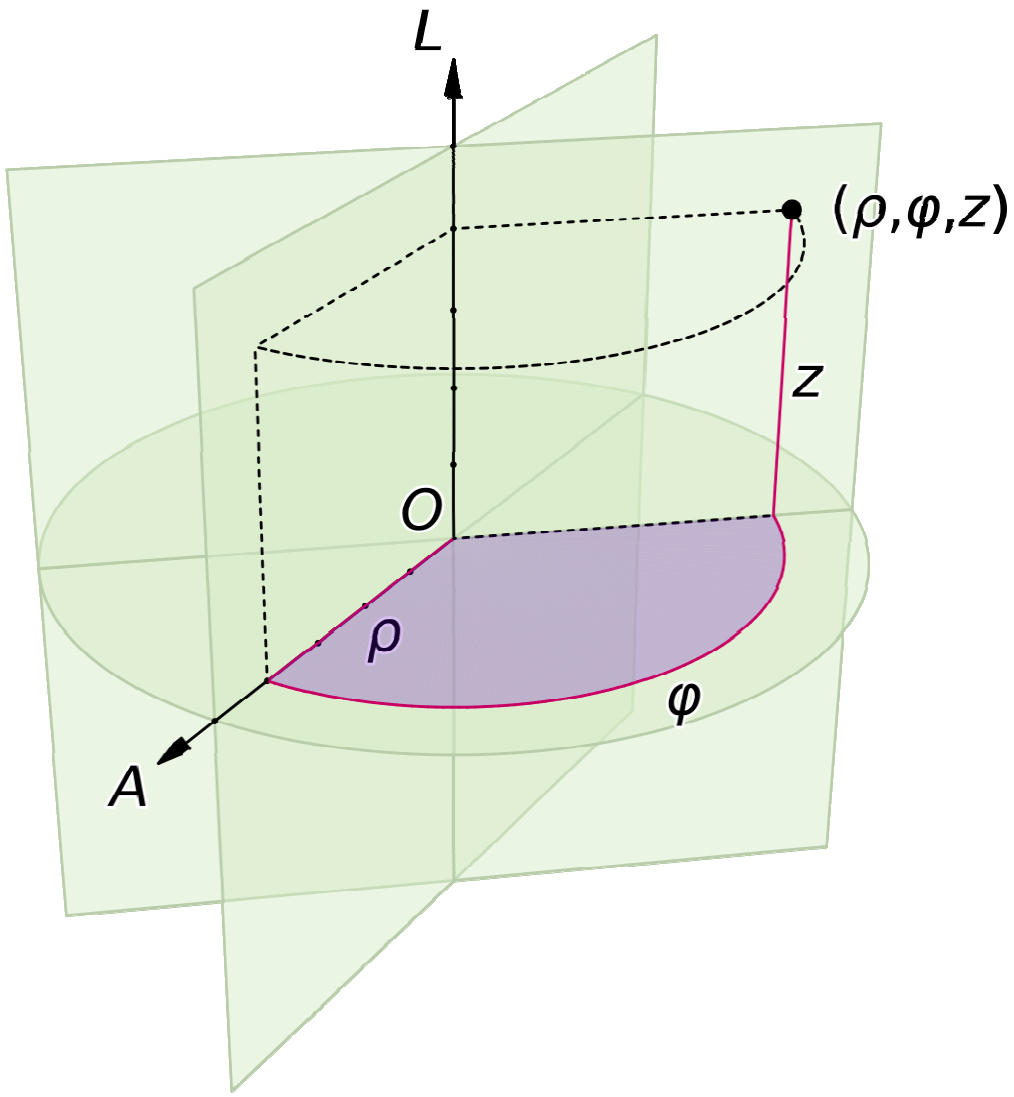
LAPLACE'S EQUATION IN CYLINDRICAL POLAR CO-ORDINATES

Transformation of coordinates from Cartesian to Cylindrical polar coordinates, of a given function let's say Ψ, (that is $\Psi(x, y, z) \rightarrow \Psi(\rho,\ \varphi,\ z)$) is governed by the following fundamental relationships:

$$x = \rho cos\varphi$$

$$y = \rho r sin\varphi$$

$$z = z$$



The laplacian in Cartesian coordinates is given by the expression

$$\nabla_c^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Now our task here is to find a corresponding expression for "∇_c^2" under the above coordinates transformation. Since the transformation affects only on the horizontal components x and y but not the vertical, the rate of change with respect to z is zero, i.e.

$$\frac{\partial}{\partial z} = 0 \Longrightarrow \frac{\partial^2}{\partial z^2} = 0$$

The partial derivatives in Cartesian coordinates with respect to the cylindrical ones are:

$$\frac{\partial x}{\partial \rho} = cos\varphi; \ \frac{\partial y}{\partial \rho} = sin\varphi$$

$$\frac{\partial x}{\partial \varphi} = -\rho sin\varphi;\ \frac{\partial y}{\partial \varphi} = \rho cos\varphi$$

Moreover;

$$\frac{\partial}{\partial \rho} = \frac{\partial x}{\partial \rho}.\frac{\partial}{\partial x} + \frac{\partial y}{\partial \rho}.\frac{\partial}{\partial y}$$

$$\Longrightarrow \frac{\partial}{\partial \rho} = cos\varphi \frac{\partial}{\partial x} + sin\varphi \frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \varphi} = -\rho sin\varphi \frac{\partial}{\partial x} + \rho cos\varphi\ \frac{\partial}{\partial y}$$

In a matrix form, the above two equations can be written as

$$\begin{pmatrix} cos\varphi & sin\varphi \\ -\rho sin\varphi & \rho cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}$$

From this matrix therefore, we need to solve for $\partial/\partial x$ and $\partial/\partial y$ in terms of $\partial/\partial \rho$ and $\partial/\partial \varphi$ using the determinant method

Therefore let

$$A = \begin{pmatrix} cos\varphi & sin\varphi \\ -\rho sin\varphi & \rho cos\varphi \end{pmatrix}$$

Then

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = A^{-1} \begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}$$

where A^{-1} is the inverse matrix of A. Now, to begin with, let's find A^{-1}

$$A = \begin{pmatrix} cos\varphi & sin\varphi \\ -\rho sin\varphi & \rho cos\varphi \end{pmatrix} \Longrightarrow A^{-1} = \frac{\begin{pmatrix} \rho cos\varphi & -sin\varphi \\ \rho sin\varphi & cos\varphi \end{pmatrix}}{Det(A)}$$

$$Det(A) = (\rho cos\varphi)(cos\varphi) - (-\rho sin\varphi)(sin\varphi) = \rho cos^2\varphi + \rho sin^2\varphi = \rho$$

Therefore

$$A^{-1} = \begin{pmatrix} cos\varphi & -\dfrac{sin\varphi}{\rho} \\ sin\varphi & \dfrac{cos\varphi}{\rho} \end{pmatrix}$$

A quick check to confirm if the above matrix is an inverse matrix of A.

Two matrices B and B^{-1} are inverses of each other if

$$\boldsymbol{B^{-1}B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I}$$

where I is the identity matrix

Hence

$$\boldsymbol{A^{-1}A = \begin{pmatrix} cos\varphi & -\dfrac{sin\varphi}{\rho} \\ sin\varphi & \dfrac{cos\varphi}{\rho} \end{pmatrix} \begin{pmatrix} cos\varphi & sin\varphi \\ -\rho sin\varphi & \rho cos\varphi \end{pmatrix}}$$

$$= \begin{pmatrix} cos^2\varphi + \left(-\dfrac{sin\varphi}{\rho}\right)(-\rho sin\varphi) & cos\varphi sin\varphi + \left(-\dfrac{sin\varphi}{\rho}\right)(\rho cos\varphi) \\ sin\varphi cos\varphi + \left(\dfrac{cos\varphi}{\rho}\right)(-\rho sin\varphi) & sin\varphi sin\varphi + \left(\dfrac{cos\varphi}{\rho}\right)(\rho cos\theta) \end{pmatrix}$$

$$= \begin{pmatrix} cos^2\varphi + sin^2\varphi & cos\varphi sin\varphi - cos\varphi sin\varphi \\ sin\varphi cos\varphi - sin\varphi cos\varphi & sin^2\varphi + cos^2\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore it's correct that the above expression for A^{-1} is an inverse matrix of the original matrix A

Eventually;

$$\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} = A^{-1} \begin{pmatrix} \dfrac{\partial}{\partial \rho} \\ \dfrac{\partial}{\partial \varphi} \end{pmatrix} \Rightarrow \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} cos\varphi & -\dfrac{sin\varphi}{\rho} \\ sin\varphi & \dfrac{cos\varphi}{\rho} \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial \rho} \\ \dfrac{\partial}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} cos\varphi \dfrac{\partial}{\partial \rho} - \dfrac{sin\varphi}{\rho}\dfrac{\partial}{\partial \varphi} \\ sin\varphi \dfrac{\partial}{\partial \rho} + \dfrac{cos\varphi}{\rho}\dfrac{\partial}{\partial \varphi} \end{pmatrix}$$

Finally

$$\frac{\partial}{\partial x} = cos\varphi \frac{\partial}{\partial \rho} - \frac{sin\varphi}{\rho}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = sin\varphi \frac{\partial}{\partial \rho} + \frac{cos\varphi}{\rho}\frac{\partial}{\partial \varphi}$$

Then

$$\frac{\partial}{\partial x} = cos\varphi \frac{\partial}{\partial \rho} - \frac{sin\varphi}{\rho}\frac{\partial}{\partial \varphi} \Rightarrow \frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left[cos\varphi \frac{\partial}{\partial \rho} - \frac{sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = cos\varphi \frac{\partial}{\partial \rho}\left[cos\varphi \frac{\partial}{\partial \rho} - \frac{sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\right] - \frac{sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\left[cos\varphi \frac{\partial}{\partial \rho} - \frac{sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\right]$$

$$= cos\varphi\left[cos\varphi \frac{\partial^2}{\partial \rho^2} - sin\varphi\left(\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\right)\frac{\partial}{\partial \varphi} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \varphi}\right)\right)\right]$$

$$- \frac{sin\varphi}{\rho}\left[\frac{\partial}{\partial \varphi}(cos\varphi)\frac{\partial}{\partial \rho} + cos\varphi\frac{\partial}{\partial \varphi}\left(\frac{\partial}{\partial \rho}\right) - \frac{1}{\rho}\left(\frac{\partial}{\partial \varphi}(sin\varphi)\frac{\partial}{\partial \varphi} + sin\varphi\frac{\partial}{\partial \varphi}\left(\frac{\partial}{\partial \varphi}\right)\right)\right]$$

$$= cos\varphi\left[cos\varphi \frac{\partial^2}{\partial \rho^2} - sin\varphi\left(-\frac{1}{\rho^2}\frac{\partial}{\partial \varphi} + \frac{1}{\rho}\frac{\partial^2}{\partial \rho \partial \varphi}\right)\right]$$

$$- \frac{sin\varphi}{\rho}\left[-sin\varphi\frac{\partial}{\partial \rho} + cos\varphi\frac{\partial^2}{\partial \rho \partial \varphi} - \frac{1}{\rho}\left(cos\varphi\frac{\partial}{\partial \varphi} + sin\varphi\frac{\partial^2}{\partial \varphi^2}\right)\right]$$

$$= \left[\cos^2\varphi\frac{\partial^2}{\partial \rho^2} + \frac{cos\varphi sin\varphi}{\rho^2}\frac{\partial}{\partial \varphi} - \frac{cos\varphi sin\varphi}{\rho}\frac{\partial^2}{\partial \rho \partial \varphi}\right]$$

$$+ \left[\frac{\sin^2\varphi}{\rho}\frac{\partial}{\partial \rho} - \frac{cos\varphi sin\varphi}{\rho}\frac{\partial^2}{\partial r \partial \varphi} + \frac{cos\varphi sin\varphi}{\rho^2}\frac{\partial}{\partial \varphi} + \frac{\sin^2\varphi}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \cos^2\varphi\frac{\partial^2}{\partial \rho^2} + \frac{\sin^2\varphi}{\rho^2}\frac{\partial^2}{\partial \varphi^2} - \frac{2cos\varphi sin\varphi}{\rho}\frac{\partial^2}{\partial \rho \partial \varphi} + \frac{\sin^2\varphi}{\rho}\frac{\partial}{\partial \rho} + \frac{2cos\varphi sin\varphi}{\rho^2}\frac{\partial}{\partial \varphi}$$

Similarly;

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}\left[sin\varphi\frac{\partial}{\partial \rho} + \frac{cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\right]$$

$$\Rightarrow \frac{\partial^2}{\partial y^2} = sin\varphi\frac{\partial}{\partial \rho}\left[sin\varphi\frac{\partial}{\partial \rho} + \frac{cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\right] + \frac{cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\left[sin\varphi\frac{\partial}{\partial \rho} + \frac{cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\right]$$

$$= sin\varphi\left[sin\varphi\frac{\partial^2}{\partial \rho^2} + cos\varphi\left(\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\right)\frac{\partial}{\partial \varphi} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \varphi}\right)\right)\right]$$

$$+ \frac{cos\varphi}{\rho}\left[\frac{\partial}{\partial \varphi}(sin\varphi)\frac{\partial}{\partial \rho} + sin\varphi\frac{\partial}{\partial \varphi}\left(\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho}\left(\frac{\partial}{\partial \varphi}(cos\varphi)\frac{\partial}{\partial \varphi} + cos\varphi\frac{\partial}{\partial \varphi}\left(\frac{\partial}{\partial \varphi}\right)\right)\right]$$

$$sin\varphi\left[sin\varphi\frac{\partial^2}{\partial \rho^2} + cos\varphi\left(-\frac{1}{\rho^2}\frac{\partial}{\partial \varphi} + \frac{1}{\rho}\frac{\partial^2}{\partial \rho \partial \varphi}\right)\right]$$

$$+ \frac{cos\varphi}{\rho}\left[cos\varphi\frac{\partial}{\partial \rho} + sin\varphi\frac{\partial^2}{\partial \rho \partial \varphi} + \frac{1}{\rho}\left(-sin\varphi\frac{\partial}{\partial \varphi} + cos\varphi\frac{\partial^2}{\partial \varphi^2}\right)\right]$$

$$\left[\sin^2\varphi\frac{\partial^2}{\partial\rho^2}-\frac{sin\varphi cos\varphi}{\rho^2}\frac{\partial}{\partial\varphi}+\frac{sin\varphi cos\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi}\right]$$

$$+\left[\frac{\cos^2\varphi}{\rho}\frac{\partial}{\partial\rho}+\frac{sin\varphi cos\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi}-\frac{sin\varphi cos\varphi}{\rho^2}\frac{\partial}{\partial\varphi}+\frac{\cos^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$\Longrightarrow\frac{\partial^2}{\partial y^2}=\sin^2\varphi\frac{\partial^2}{\partial\rho^2}+\frac{\cos^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{2sin\varphi cos\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi}+\frac{\cos^2\varphi}{\rho}\frac{\partial}{\partial\rho}-\frac{2sin\varphi cos\varphi}{\rho^2}\frac{\partial}{\partial\varphi}$$

$$\Longrightarrow\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\left(\cos^2\varphi\frac{\partial^2}{\partial\rho^2}+\frac{\sin^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2}-\frac{2cos\varphi sin\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi}+\frac{\sin^2\varphi}{\rho}\frac{\partial}{\partial\rho}\right.$$

$$\left.+\frac{2cos\varphi sin\varphi}{\rho^2}\frac{\partial}{\partial\varphi}\right)$$

$$+\left(\sin^2\varphi\frac{\partial^2}{\partial\rho^2}+\frac{\cos^2\varphi}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{2sin\varphi cos\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi}+\frac{\cos^2\varphi}{\rho}\frac{\partial}{\partial\rho}\right.$$

$$\left.-\frac{2sin\varphi cos\varphi}{\rho^2}\frac{\partial}{\partial\varphi}\right)$$

$$=(\cos^2\varphi+\sin^2\varphi)\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}(\cos^2\varphi+\sin^2\varphi)\frac{\partial^2}{\partial\varphi^2}+\left(-\frac{2cos\varphi sin\varphi}{\rho}+\frac{2cos\varphi sin\varphi}{\rho}\right)\frac{\partial^2}{\partial\rho\partial\varphi}$$

$$+\frac{1}{\rho}(\sin^2\varphi+\cos^2\varphi)\frac{\partial}{\partial\rho}+\left(\frac{2cos\varphi sin\varphi}{\rho^2}-\frac{2cos\varphi sin\varphi}{\rho^2}\right)\frac{\partial}{\partial\varphi}$$

$$\Longrightarrow\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\Longrightarrow\boldsymbol{\nabla_c^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}}$$

Hence

$$\boldsymbol{\nabla_c^2=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}}$$

MATRICES REVIEW AS APPLIED IN INTEGRAL CALCULUS

Review Task 1*: Gien the following 3x3 matrix below, determine its inverse matrix*

$$A=\begin{pmatrix}7&2&1\\0&3&-1\\-3&4&-2\end{pmatrix}$$

SOLUTION:

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix} \Longrightarrow A^T = \begin{pmatrix} 7 & 0 & -3 \\ 2 & 3 & 4 \\ 1 & -1 & -2 \end{pmatrix}$$

The co-factors, $C(A^T)$ of the transpose matrix A^T are as follows:

$$\begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 2 & 3 \end{pmatrix}$$

$$\Longrightarrow C(A^T) = \begin{pmatrix} -6+4 & -4-4 & -2-3 \\ 0-3 & -14+3 & -7-0 \\ 0+9 & 28+6 & 21-0 \end{pmatrix} = \begin{pmatrix} -2 & -8 & -5 \\ -3 & -11 & -7 \\ 9 & 34 & 21 \end{pmatrix}$$

$$\text{Adjoint}\,(A) = C(A^T) \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} -2 & -8 & -5 \\ -3 & -11 & -7 \\ 9 & 34 & 21 \end{pmatrix} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\text{Adjoint}\,(A) = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}$$

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix} \Longrightarrow \det(A) = 7 \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - 2 \begin{pmatrix} 0 & -1 \\ -3 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ -3 & 4 \end{pmatrix}$$

$$\det(A) = 7(-6+4) - 2(0-3) + (0+9) = 7(-2) - 2(-3) + 9 = -14 + 6 + 9 = 1$$

$$\det(A) = 1$$

$$A^{-1} = \frac{\text{Adjoint}(A)}{\det(A)} = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}$$

$$\Longrightarrow A^{-1} = \begin{pmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{pmatrix}$$

ALTERNATIVELY:

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$$

The co-factors, $C(A)$ of the matrix A are

$$\begin{pmatrix}3 & -1\\ 4 & -2\end{pmatrix}\begin{pmatrix}0 & -1\\ -3 & -2\end{pmatrix}\begin{pmatrix}0 & 3\\ -3 & 4\end{pmatrix}$$

$$\begin{pmatrix}2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}7 & 1\\ -3 & -2\end{pmatrix}\begin{pmatrix}7 & 2\\ -3 & 4\end{pmatrix}$$

$$\begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}7 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}7 & 2\\ 0 & 3\end{pmatrix}$$

$$\Longrightarrow C(A) = \begin{pmatrix}-6+4 & 0-3 & 0+9\\ -4-4 & -14+3 & 28+6\\ -2-3 & -7-0 & 21-0\end{pmatrix} = \begin{pmatrix}-2 & -3 & 9\\ -8 & -11 & 34\\ -5 & -7 & 21\end{pmatrix}$$

$$\big(C(A)\big)^T = \begin{pmatrix}-2 & -8 & -5\\ -3 & -11 & -7\\ 9 & 34 & 21\end{pmatrix}$$

$$\text{Adjoint }(A) = \big(C(A)\big)^T\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix} = \begin{pmatrix}-2 & -8 & -5\\ -3 & -11 & -7\\ 9 & 34 & 21\end{pmatrix}\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix} = \begin{pmatrix}-2 & 8 & -5\\ 3 & -11 & 7\\ 9 & -34 & 21\end{pmatrix}$$

$$\text{Adjoint }(A) = \begin{pmatrix}-2 & 8 & -5\\ 3 & -11 & 7\\ 9 & -34 & 21\end{pmatrix}$$

$$\Longrightarrow A^{-1} = \begin{pmatrix}-2 & 8 & -5\\ 3 & -11 & 7\\ 9 & -34 & 21\end{pmatrix}$$

Which is a solution similar to the one found earlier, indicating that $C(A^T) = \big(C(A)\big)^T$

$$\begin{pmatrix}3 & -1\\ 4 & -2\end{pmatrix}\begin{pmatrix}0 & -1\\ -3 & -2\end{pmatrix}\begin{pmatrix}0 & 3\\ -3 & 4\end{pmatrix}$$

$$\begin{pmatrix}2 & 1\\ 4 & -2\end{pmatrix}\begin{pmatrix}7 & 1\\ -3 & -2\end{pmatrix}\begin{pmatrix}7 & 2\\ -3 & 4\end{pmatrix}$$

$$\begin{pmatrix}2 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}7 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}7 & 2\\ 0 & 3\end{pmatrix}$$

Review Task 2*: Solve for x, y and z in the following system of equations*

$$2x - 3y + z = u$$

$$x + 2y = v$$

$$x - z = w$$

In matrix notation, the above system of equations can be represented as

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Let

$$B = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \Longrightarrow B^T = \begin{pmatrix} 2 & 1 & 1 \\ -3 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

The co-factors, $C(B^T)$ of the matrix B^T are

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$\Longrightarrow C(B^T) = \begin{pmatrix} -2 & 3 & -2 \\ -1 & -3 & -1 \\ -2 & 3 & 7 \end{pmatrix}$$

$$\text{Adjoint}\,(B) = C(B^T)\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} -2 & 3 & -2 \\ -1 & -3 & -1 \\ -2 & 3 & 7 \end{pmatrix}\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} -2 & -3 & -2 \\ 1 & -3 & 1 \\ -2 & -3 & 7 \end{pmatrix}$$

$$\text{Adjoint}\,(B) = \begin{pmatrix} -2 & -3 & -2 \\ 1 & -3 & 1 \\ -2 & -3 & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \Longrightarrow \det(B) = 2\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$= 2(-2) + 3(-1) + (0 - 2) = -4 - 3 - 2 = -9$$

$$\det(B) = -9$$

$$B^{-1} = \frac{\text{Adjoint}(B)}{\det(B)} = -\frac{1}{9}\begin{pmatrix} -2 & -3 & -2 \\ 1 & -3 & 1 \\ -2 & -3 & 7 \end{pmatrix} == \frac{1}{9}\begin{pmatrix} 2 & 3 & 2 \\ -1 & 3 & -1 \\ 2 & 3 & -7 \end{pmatrix}$$

$$B^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 3 & 2 \\ -1 & 3 & -1 \\ 2 & 3 & -7 \end{pmatrix}$$

Note that

$B^{-1}B = BB^{-1} = I$; where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Review Task 3*: A quick check to see if this is true for the above matrix B*

$$B = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix};\ B^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 3 & 2 \\ -1 & 3 & -1 \\ 2 & 3 & -7 \end{pmatrix} \Rightarrow BB^{-1} = \frac{1}{9}\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ -1 & 3 & -1 \\ 2 & 3 & -7 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} (2)2 + (-3)(-1) + 1(2) & (2)3 + (-3)(3) + 1(3) & (2)2 + (-3)(-1) + 1(-7) \\ 1(2) + 2(-1) + 0 & 1(3) + 2(3) + 0 & 1(2) + 2(-1) + 0 \\ 1(2) + 0 + (-1)(2) & 1(3) + 0 + (-1)(3) & 1(2) + 0 + (-1)(-7) \end{pmatrix}$$

$$BB^{-1} = \frac{1}{9}\begin{pmatrix} 4 + 3 + 2 & 6 - 9 + 3 & 4 + 3 - 7) \\ 2 - 2 & 3 + 6 & 2 - 2 \\ 2 - 2 & 3 - 3 & 2 + 7) \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0) \\ 0 & 9 & 0 \\ 0 & 0 & 9) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} = I$$

Therefore

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \Rightarrow \frac{1}{9}\begin{pmatrix} 2 & 3 & 2 \\ -1 & 3 & -1 \\ 2 & 3 & -7 \end{pmatrix}\begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 2 & 3 & 2 \\ -1 & 3 & -1 \\ 2 & 3 & -7 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2u + 3v + 2w \\ -u + 3v - w \\ 2u + 3v - 7w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Therefore the solution to the system of equations

$$u = 2x - 3y + z$$

$$v = x + 2y$$

$$w = x - z$$

is

$$x = \frac{1}{9}(2u + 3v + 2w)$$

$$y = \frac{1}{9}(-u + 3v - w)$$

$$z = \frac{1}{9}(2u + 3v - 7w)$$

and the Jacobean is

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \left| \begin{pmatrix} \frac{2}{9} & \frac{1}{3} & \frac{2}{9} \\ -\frac{1}{9} & \frac{1}{3} & -\frac{1}{9} \\ \frac{2}{9} & \frac{1}{3} & -\frac{7}{9} \end{pmatrix} \right|$$

$$= \left| \frac{2}{9} \cdot \begin{pmatrix} \frac{1}{3} & -\frac{1}{9} \\ \frac{1}{3} & -\frac{7}{9} \end{pmatrix} - \frac{1}{3} \cdot \begin{pmatrix} -\frac{1}{9} & -\frac{1}{9} \\ \frac{2}{9} & -\frac{7}{9} \end{pmatrix} + \frac{2}{9} \cdot \begin{pmatrix} -\frac{1}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{1}{3} \end{pmatrix} \right|$$

$$= \left| \frac{2}{9}\left(-\frac{7}{27} + \frac{1}{27}\right) - \frac{1}{3}\left(\frac{7}{81} + \frac{2}{81}\right) + \frac{2}{9}\left(-\frac{1}{27} - \frac{2}{27}\right) \right|$$

$$= \left| \frac{2}{9}\left(-\frac{6}{27}\right) - \frac{1}{3}\left(\frac{9}{81}\right) + \frac{2}{9}\left(-\frac{1}{9}\right) \right| = \left| -\frac{4}{81} - \frac{3}{81} - \frac{2}{81} \right| = \left| -\frac{9}{81} \right| = \left| -\frac{1}{9} \right| = \frac{1}{9}$$

Review Task 4**:** Find D^{-1} given that

$$D = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

SOLUTION:

$$D = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow D^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathrm{C}(D^T) =$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\Longrightarrow C(D^T) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Adjoint}\,(D) = C(D^T)\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{Adjoint}\,(D) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow \det(D) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 - 0 + 0 = 1$$

$$\det(D) = 1$$

$$D^{-1} = \frac{\text{Adjoint}(D)}{\det(D)} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore

$$D^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Check:

$$DD^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Review Task 5: Now consider the following system of equations

$$x + y + z = u$$

$$y + z = v$$

$$z = w$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u - v \\ v - w \\ w \end{pmatrix}$$

Therefore we observe from above that

$x = u - v; y = v - w; z = w$ or

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}u\\ v\\ w\end{pmatrix}$$

$$u = x + y + z$$

$$v = y + z$$

$$z = w$$

$$\boldsymbol{x = u - v}$$

$$\boldsymbol{y = v - w}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{vmatrix}=\left|1.\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}-(-1).\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}+0.\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right|$$
$$= |1 + 0 + 0| = 1$$

<u>Review Task 6</u>:

Let

$$A=\begin{pmatrix}4 & 2\\ 16 & -1\end{pmatrix}$$

$$\text{Determinat }(A) = \text{Det }(A) = 4(-1) - 16(2) = -4 - 32 = -36$$

$$\text{Det }(A) = -36$$

$$A^{-1}=\frac{-1}{36}\begin{pmatrix}-1 & -2\\ -16 & 4\end{pmatrix}=\frac{1}{36}\begin{pmatrix}1 & 2\\ 16 & -4\end{pmatrix}$$

$$\Longrightarrow A^{-1}=\frac{1}{36}\begin{pmatrix}1 & 2\\ 16 & -4\end{pmatrix}$$

$$A^{-1}A=\frac{1}{36}\begin{pmatrix}1 & 2\\ 16 & -4\end{pmatrix}\begin{pmatrix}4 & 2\\ 16 & -1\end{pmatrix}=\frac{1}{36}\begin{pmatrix}4+32 & 2-2\\ 64-64 & 32+4\end{pmatrix}=\frac{1}{36}\begin{pmatrix}36 & 0\\ 0 & 36\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Therefore, for the system of equations

$$\begin{pmatrix} 4 & 2 \\ 16 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix}$$

We can determine the values of the variables x and y as follows:

$$\begin{pmatrix} 4 & 2 \\ 16 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix} \Rightarrow \frac{1}{36}\begin{pmatrix} 1 & 2 \\ 16 & -4 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 16 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{36}\begin{pmatrix} 1 & 2 \\ 16 & -4 \end{pmatrix}\begin{pmatrix} 8 \\ 14 \end{pmatrix}$$
$$= \frac{1}{36}\begin{pmatrix} 8+28 \\ 128-56 \end{pmatrix} = \frac{1}{36}\begin{pmatrix} 36 \\ 72 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ or } (x, y) = (1, 2)$$

Review Task 7:

Similarly, for a simultaneous equation

$$3x + 2y = 7$$

$$-6x + 6y = 6$$

or (in matrix notation)

$$\begin{pmatrix} 3 & 2 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

Let

$$A = \begin{pmatrix} 3 & 2 \\ -6 & 6 \end{pmatrix}$$

$$\text{Determinat } (A) = \text{Det } (A) = 3(6) + 6(2) = 30$$

$$\Rightarrow A^{-1} = \frac{1}{30}\begin{pmatrix} 6 & -2 \\ 6 & 3 \end{pmatrix}$$

Check to see if this value of the matrix A^{-1} is correct

$$A^{-1}A = \frac{1}{30}\begin{pmatrix} 6 & -2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -6 & 6 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 18+12 & 2-2 \\ 18-18 & 12+18 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 30 & 0 \\ 0 & 30 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 6 & -2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 6 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 42-12 \\ 42+18 \end{pmatrix} = \frac{1}{30}\begin{pmatrix} 30 \\ 60 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow (x, y) = (1, 2)$$

Example 1.

Using integral calculus, find the volume of the parallelepiped defined by the inequalities:

$$0 \leq 2x - 3y + z \leq 5$$

$$1 \leq x + 2y \leq 4$$

$$-3 \leq x - z \leq 6$$

Let

$$u = 2x - 3y + z \qquad (i)$$

$$v = x + 2y \qquad (ii)$$

$$w = x - z \qquad (iii)$$

$\Longrightarrow$

$$0 \leq u \leq 5$$

$$1 \leq v \leq 4$$

$$-3 \leq w \leq 6$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Longrightarrow J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = 2\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} - (-3)\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + 1\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$J = 2(-2) + 3(-1) + 1(-2) = -9$$

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1} dudvdw$$

$$|J|^{-1} = |-9J|^{-1} = \frac{1}{9} \Longrightarrow dxdydz = \frac{1}{9} dudvdw$$

Therefore

$$V = \iiint \frac{1}{9} dudvdw = \frac{1}{9}\int_0^5 du \int_1^4 dv \int_{-3}^6 dw = \frac{1}{9}(5)(4-1)(6+3)$$

$$V = \frac{1}{9}(5)(3)(9) = 15 \; cubic \; units$$

Note:

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1} = |J|^{-1} = |-9J|^{-1} = \frac{1}{9}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{9} & \frac{3}{9} & \frac{2}{9} \\ -\frac{1}{9} & \frac{3}{9} & -\frac{1}{9} \\ \frac{2}{9} & \frac{3}{9} & -\frac{7}{9} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{9^3} \begin{pmatrix} 2 & 3 & 2 \\ -1 & 3 & -1 \\ 2 & 3 & -7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$J = \frac{2}{9}\begin{pmatrix} \frac{3}{9} & -\frac{1}{9} \\ \frac{3}{9} & -\frac{7}{9} \end{pmatrix} - \frac{3}{9}\begin{pmatrix} -\frac{1}{9} & -\frac{1}{9} \\ \frac{2}{9} & -\frac{7}{9} \end{pmatrix} + \frac{2}{9}\begin{pmatrix} -\frac{1}{9} & \frac{3}{9} \\ \frac{2}{9} & \frac{3}{9} \end{pmatrix}$$

$$J = \frac{2}{9}\left[\left(\frac{3}{9}\right)\left(-\frac{7}{9}\right) - \left(\frac{3}{9}\right)\left(-\frac{1}{9}\right)\right] - \frac{3}{9}\left[\left(-\frac{1}{9}\right)\left(-\frac{7}{9}\right) - \left(\frac{2}{9}\right)\left(-\frac{1}{9}\right)\right] + \frac{2}{9}\left[\left(-\frac{1}{9}\right)\left(\frac{3}{9}\right) - \left(\frac{2}{9}\right)\left(\frac{3}{9}\right)\right]$$

$$J = \frac{2}{9}\left[-\frac{21}{81} + \frac{3}{81}\right] - \frac{3}{9}\left[\frac{7}{81} + \frac{2}{81}\right] + \frac{2}{9}\left[-\frac{3}{81} - \frac{6}{81}\right]$$

$$J = \frac{2}{9}\left[-\frac{18}{81}\right] - \frac{3}{9}\left[\frac{9}{81}\right] + \frac{2}{9}\left[-\frac{9}{81}\right] = -\frac{4}{81} - \frac{3}{81} - \frac{2}{81} = -\frac{9}{81} = -\frac{1}{9}$$