

ADVANCED MATHEMATICS

INTEGRATION

10

$$V = \pi \int_{-r}^r (r^2 - x^2) dx = \frac{4}{3} \pi r^3$$

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To myself and I

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INTEGRATION

Chapter

10

Integration is process of reversing differentiation. Previously, we learned differentiation of different relations. Here we are going to find out the relation/function whenever the derivative is given. The required relation/function is called anti – derivative or definite integrals.



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The origin of integral calculus can be traced back to the ancient Greeks. They were motivated by the need to measure the length of curves, the area of a surface, or the volume of a solid. Archimedes used techniques very similar to actual integration to determine the length of a segment of a curve. *Democritus 410 B.C.* had the insight to consider that a cone was made up of infinitely many planes cross sections parallel to the base.

The theory of integration received very little stimulus after Archimedes' remarkable achievements. It was not until the beginning of the 17th century that the interest in Archimedes' ideas began to develop. *Johann Kepler 1571 – 1630 A.D.* was the first among European mathematicians to develop the ideas of infinitesimals in connection with integration. The use of the term "integral" is due to the Swiss mathematician *Johann Bernoulli 1667 – 1748*. In the present chapter, we shall study integration of real-valued functions of a single variable according to the concepts put forth by the German mathematician *Georg Friedrich Riemann 1826 – 1866*. He was the first to establish a rigorous analytical foundation for integration, based on the older geometric approach.

Suppose we are given the function $y = 3x^2 + 4x - 3$, it is the derivative $\frac{dy}{dx} = 6x + 4$. The question is can do reverse to find y ? It is possible to get y , the rule that guide integration

is $\int x^n dx = \frac{x^{n+1}}{n+1} + A$ where A is a constant of integration and $n \neq -1$. So for the

derivative $\frac{dy}{dx} = 6x + 4$ multiply both sides dx

$$dx \frac{dy}{dx} = (6x + 4)dx \Rightarrow dy = (6x + 4)dx$$

Put the integral both sides, as $\int dy = \int (6x + 4)dx$

$$\int dy = \int 3x dx + \int 4 dx \text{ thus } y = 3 \int x dx + 4 \int dx \text{ therefore } y = \frac{3}{2}x^2 + 4x + A$$

Indefinite integrals

The indefinite integrals: These are integrals of the form $\int f(x)dx$ thus if

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

Example 1

Evaluate the integral $\int (2x^3 + 4x^2 + 7)dx$

Solution

From $\int (2x^3 + 4x^2 + 7)dx$

$$\int (2x^3 + 4x^2 + 7)dx = 2 \int x^3 dx + 4 \int x^2 dx + 7 \int dx$$

$$2 \int x^3 dx + 4 \int x^2 dx + 7 \int dx \text{ thus } = 2 \left(\frac{x^4}{4} \right) + 4 \left(\frac{x^3}{3} \right) + 7x + A$$

$$\therefore \int (2x^3 + 4x^2 + 7)dx = \frac{x^4}{2} + \frac{4x^3}{3} + 7x + A$$

Example 2

Evaluate $\int (2x - 3)dx$

Solution

From $\int (2x - 3)dx = \int 2x dx - \int 3 dx$

$$2 \int x dx - 3 \int dx \text{ thus } \int (2x - 3)dx = x^2 - 3x + A$$

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Example 3

Evaluate $\int \left(3x^2 - 5x + \frac{6}{x^3} - \frac{1}{x} \right) dx$

Solution

$$\int (3x^2 - 5x) dx = 3 \int x^2 dx - 5 \int x dx \text{ thus } = 3 \left(\frac{x^3}{3} \right) - 5 \left(\frac{x^2}{2} \right) + A$$

$$\therefore \int (3x^2 - 5x) dx = x^3 - \frac{5x^2}{2} + A$$

Example 4

Evaluate $\int (2 - 3x + 5x^3) dx$

Solution

$$\begin{aligned} \text{From } \int (2 - 3x + 5x^3) dx &= 2 \int dx - 3 \int x dx + 5 \int x^3 dx \\ &= 2x - 3 \left(\frac{x^2}{2} \right) + 5 \left(\frac{x^4}{4} \right) + A \end{aligned}$$

$$\text{Therefore } \int (2 - 3x + 5x^3) dx = 2x - \frac{3x^2}{2} + \frac{5x^4}{4} + A$$

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Example 5

Evaluate $\int (x^3 - 3x^2) dx$

Solution

$$\text{Given } \int (x^3 - 3x^2) dx \text{ thus } \int (x^3 - 3x^2) dx = \int x^3 dx - 3 \int x^2 dx$$

$$\therefore \int (x^3 - 3x^2) dx = \frac{x^4}{4} - x^3 + A$$

EXERCISE 1

1. $\int (x^3 - 2x^2 + 5) dx$

2. $\int (x^3 - 2x^2)(x - 2) dx$

3. $\int (4x^3 - 5x + 2) dx$

4. $\int \left(3 - \frac{3}{x^3} + 5x^4 + 12x \right) dx$

5. $\int (4 - 3x + 10x^4 - 2x^3) dx$

6. $\int \left(\frac{1}{6}x^3 - 3x^2 + \frac{1}{2}x \right) dx$

7. $\int (3x^2 - 4x) dx$

8. $\int \left(x^3 - \frac{1}{x^2} \right) dx$

9. $\int (x^4 - 2x^3 + x^2 - x) dx$

10. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} + \sqrt[3]{x} \right) dx$

11. $\int \left(2\sqrt{x} - \frac{1}{3\sqrt[5]{x}} \right) dx$

12. $\int \left(\frac{x}{5\sqrt{x}} - \frac{4}{\sqrt[3]{x}} \right) dx$

Substitution / Change of variable

Integrals of the form $\int f(x)f'(x)dx$ are easily evaluated by letting

$$u = f(x) \Rightarrow \frac{du}{dx} = f'(x) \text{ hence } dx = \frac{du}{f'(x)}$$

Example 6

Evaluate the integral $\int (2x + 3)(x^2 + 3x - 2) dx$

Solution

From $\int (2x + 3)(x^2 + 3x - 2) dx$

Let $u = x^2 + 3x - 2$ then $\frac{du}{dx} = 2x + 3$

Make dx the subject and substitute

$$dx = \frac{du}{(2x + 3)} \text{ then } \int (2x + 3)u \frac{du}{(2x + 3)}$$

$$\int u du = \left(\frac{u^2}{2} \right) + A$$

$$\text{But } u = x^2 + 3x - 2$$

$$\int (2x+3)(x^2+3x-2)dx = \frac{1}{2}(x^2+3x-2)^2 + A$$

Example 7

$$\text{Evaluate } \int (2x-6)(2x^2-12x)dx$$

Solution

$$\text{From } \int (2x-6)(2x^2-12x)dx, \text{ Let } u = 2x^2 - 12x \text{ and } \frac{du}{dx} = 4x - 12$$

$$dx = \frac{du}{2(2x-6)} \text{ and } \int (2x-6)u \frac{du}{2(2x-6)}$$

$$\frac{1}{2} \int u du = \frac{1}{2} \left(\frac{u^2}{2} \right) + A \quad \text{But } u = 2x^2 - 12x$$

$$\therefore \int (2x-6)(2x^2-12x)dx = \frac{1}{4}(2x^2-12x)^2 + A$$

Example 8

$$\text{Evaluate } \int (3x^2+1)\sqrt{5x^3+5x}dx$$

Solution

$$\text{Let } u = \sqrt{5x^3+5x} \Rightarrow u^2 = 5x^3+5x$$

$$2u \frac{du}{dx} = 15x^2 + 5 \text{ Substituting}$$

$$dx = \frac{2udu}{5(3x^2+1)} \text{ and } \int (3x^2+1)u \frac{2udu}{5(3x^2+1)}$$

$$\frac{2}{5} \int u^2 du = \frac{2}{15} u^3 + A \quad \text{But } u = \sqrt{5x^3+5x}$$

$$\int (3x^2+1)\sqrt{5x^3+5x}dx = \frac{1}{15} \left(\sqrt{5x^3+5x} \right)^3 + A$$

$$\text{Therefore, } \int (3x^2+1)\sqrt{5x^3+5x}dx = \frac{1}{15} (5x^3+5x)\sqrt{5x^3+5x} + A$$

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Example 9

Evaluate $\int (7x+1)(7x^2+2x-1)^5 dx$

Solution

Given $\int (7x+1)(7x^2+2x-1)^5 dx$ Let $u = 7x^2 + 2x - 1$ therefore

$$dx = \frac{du}{2(7x+1)} \text{ thus } \int (7x+1)u^5 \frac{du}{2(7x+1)}$$

$$\frac{1}{2} \int u^5 du = \frac{1}{12} u^6 + A$$

$$\text{But } u = 7x^2 + 2x - 1$$

$$\text{Therefore, } \int (7x+1)(7x^2+2x-1)^5 dx = \frac{1}{12} (7x^2+2x-1)^6 + A$$

Example 10

Evaluate $\int (2-3x)(3x^2-4x)^9 dx$

Solution

$$\int (2-3x)(3x^2-4x)^9 dx \quad \text{Let } u = 3x^2 - 4x \text{ then } \frac{du}{dx} = 6x - 4$$

$$du = -\frac{du}{2(2-3x)} \text{ then } -\int (2-3x)u^9 \frac{du}{2(2-3x)}, \text{ But } u = 3x^2 - 4x$$

$$-\frac{1}{2} \int u^9 du = -\frac{1}{20} (3x^2 - 4x)^{10} + A$$

$$\therefore \int (2-3x)(3x^2-4x)^9 dx = -\frac{1}{20} (3x^2-4x)^{10} + A$$

EXERCISE 2

Evaluate the following

1. $\int (6x-1)(3x^2-x+2)dx$

2. $\int (2x-1)\sqrt[3]{4x^2-4x+3}dx$

3. $\int \sin 3x \sqrt{\cos 3x} dx$

4. $\int \sin^7 x \cos x dx$

5. $\int \cos^3 x dx$

6. $\int \sin 5x \cos 3x dx$

7. $\int \cos x \sin^3 x dx$

8. $\int \sin^2 4x \cos 4x dx$

Solution

From $\int \frac{dx}{2x+1}$ Let $u = 2x+1$

$$\frac{du}{dx} = 2 \qquad dx = \frac{du}{2}$$

$$\int \frac{1}{u} \left(\frac{du}{2} \right) \text{ thus } \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + A, \quad \text{But } u = 2x+1$$

$$\therefore \int \frac{dx}{2x+1} = \frac{1}{2} \ln|2x+1| + A$$

Example 12

Evaluate $\int \frac{x}{3x^2+9} dx$

Solution

$\int \frac{x}{3x^2+9} dx$ Let $u = 3x^2 + 9$

$$\frac{du}{dx} = 6x \qquad dx = \frac{du}{6x}$$

Substituting $\int \left(\frac{x}{u} \right) \left(\frac{du}{6x} \right)$

$$\frac{1}{6} \int \frac{du}{u} = \frac{1}{6} \ln|3x^2+9| + A \qquad \text{But } u = 3x^2 + 9$$

$$\therefore \int \frac{x}{3x^2+9} dx = \frac{1}{6} \ln|3x^2+9| + A$$

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Example 13

Evaluate $\int \frac{x^2+3x-1}{\sqrt{x}} dx$

Solution

$$\int \frac{x^2+3x-1}{\sqrt{x}} dx \text{ simplifying } \int \left(\frac{x^2}{x^{\frac{1}{2}}} + \frac{3x}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} \right) dx$$

$$\int (x^{1.5}) dx + 3 \int x^{0.5} dx - \int x^{-0.5} dx$$

$$= \frac{5x^{\frac{5}{2}}}{2} + 2x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + A$$

$$\therefore \int \left(\frac{x^2 + 3x - 1}{\sqrt{x}} \right) dx = \frac{2}{5} x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + A$$

Example 14

Evaluate $\int \frac{\cos x}{\sin x + 2} dx$

Solution

From $\int \frac{\cos x}{\sin x + 2} dx$

Let $u = \sin x + 2$

$$\frac{du}{dx} = \cos x \Rightarrow du = \cos x dx$$

$$\int \frac{du}{u} = \ln|u| + A$$

But $u = \sin x + 2$

$$\therefore \int \frac{\cos x}{\sin x + 2} dx = \ln|\sin x + 2| + A$$

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Example 15

Find $\int \left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \right) dx$

Solution

Let $u = e^{2x} + e^{-2x}$ thus $\frac{du}{dx} = 2(e^{2x} - e^{-2x})$

$$\frac{du}{2} = (e^{2x} - e^{-2x}) dx$$

$$\int \frac{du}{2u} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + A$$

But $u = e^{2x} + e^{-2x}$

$$\therefore \int \left(\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \right) dx = \frac{1}{2} \ln|e^{2x} + e^{-2x}| + A$$

Example 16

Find $\int \frac{2x+3}{x^2+3x-2} dx$

Solution

Let $u = x^2 + 3x - 2$ then $\frac{du}{dx} = 2x + 3 \Rightarrow dx = \frac{du}{2x+3}$

$$\int \frac{2x+3}{u} \left(\frac{du}{2x+3} \right)$$

$$\int \frac{du}{u} = \ln|u| + A, \quad \text{But } u = x^2 + 3x - 2$$

$$\therefore \int \frac{2x+3}{x^2+3x-2} dx = \ln|x^2+3x-2| + A$$

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Example 17

Find $\int \frac{3x-5}{3x^2-10x+7} dx$

Solution

From $\int \frac{3x-5}{3x^2-10x+7} dx$

$$\frac{du}{dx} = 6x - 10$$

Let $u = 3x^2 - 10x + 7$

$$dx = \frac{du}{2(3x-5)}$$

$$\int \frac{3x-5}{u} \frac{du}{2(3x-5)} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln|u| + A$$

But $u = 3x^2 - 10x + 7$

Therefore, $\int \frac{3x-5}{3x^2-10x+7} dx = \frac{1}{2} \ln|3x^2-10x+7| + A$

Example 18

Evaluate $\int \frac{x^2 - 2x + 3}{x} dx$

Solution

Separate the integrals as $\int \frac{x^2 - 2x + 3}{x} dx$ thus $\int x dx - 2 \int dx + 3 \int \frac{dx}{x}$

$$\therefore \int \frac{x^2 - 2x + 3}{x} = \frac{x^2}{2} - 2x + 3 \ln|x| + A$$

Example 19

Find $\int \frac{\cot x}{\ln|\sin x|} dx$

Solution

$$\int \frac{\cot x}{\ln|\sin x|} dx$$

$$\frac{du}{dx} = \cot x$$

$$\int \frac{\cot x}{u} \left(\frac{du}{\cot x} \right) \text{ thus } \int \frac{du}{u} = \ln|u| + A \text{ but } u = \ln|\sin x|$$

$$\therefore \int \frac{\cot x}{\ln|\sin x|} dx = \ln|\ln|\sin x|| + A$$

Let $u = \ln|\sin x|$

$$dx = \frac{du}{\cot x}$$

EXERCISE 3

Evaluate the following integrals.

1. $\int \left(\frac{\sin x}{5 + \cos x} \right) dx$

2. $\int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

3. $\int \left(\frac{\cos 3x}{1 + \sin 3x} \right) dx$

4. $\int \left(\frac{2x + 3}{x^2 + 3x - 1} \right) dx$

$$5. \int \left(\frac{\sec^2 x}{3 \tan x} \right) dx$$

$$7. \int \frac{x^2}{(x^3 + 8)^4} dx$$

$$9. \int \frac{\cos x}{\sqrt{1 + \sin x}} dx$$

$$11. \int \frac{x+2}{\sqrt{x-3}} dx$$

$$13. \int \frac{3-x}{\sqrt{1-x^2}} dx$$

$$15. \int \frac{dx}{x^2 \sqrt{1-x^2}}$$

$$17. \int \frac{x}{\sqrt{4-x^4}} dx$$

$$6. \int \frac{1}{2x+3} dx$$

$$8. \int \left(\frac{x^3}{2+x^8} \right) dx$$

$$10. \int \frac{x}{x-3} dx$$

$$12. \int \frac{2x+3}{\sqrt{4-x^2}} dx$$

$$14. \int \frac{x^2}{\sqrt{1-x^2}} dx$$

$$16. \int \frac{dx}{x^2 \sqrt{x^2-1}}$$

$$18. \int \frac{y^9}{3+y^{20}} dy$$

Standard integrals

$$1. \int x^n dx = \frac{x^{n+1}}{n+1} + A$$

$$2. \int \sin x dx = -\cos x + A$$

$$4. \int \tan x dx = -\ln|\cos x| = \ln|\sec x| + A$$

$$5. \int \cot x dx = \ln|\sin x| + A$$

$$7. \int \operatorname{cosec} x dx = -\cot x + A$$

$$9. \int \sec^2 x dx = \tan x + A$$

$$10. \int e^x dx = e^x + A$$

$$12. \int \frac{dx}{\sqrt{1-x^2}} = \cos^{-1}(x) + A$$

$$3. \int \cos x dx = \sin x + A$$

$$6. \int \sec x dx = \ln|\sec x + \tan x| + A$$

$$8. \int \sec x \tan x dx = \sec x + A$$

$$9. \int \operatorname{cosec}^2 x dx = -\cot x + A$$

$$11. \int a^x dx = \frac{a^x}{\ln a} + A, a > 0$$

$$13. \int \frac{dx}{1+x^2} = \tan^{-1}(x) + A$$

$$14. \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}(x) + A$$

$$15. \int \frac{dx}{1-x^2} = \tanh^{-1}(x) + A$$

$$16. \int \cosh x dx = \sinh x + A$$

$$17. \int \sinh x dx = \cosh x + A$$

$$18. \int \tanh x dx = \ln|\cosh x| + A$$

$$19. \int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1}(x) + A$$

$$20. \int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| + A$$

INTEGRAL OF TRIGONOMETRIC RATIOS OF THE FORM $\int \sin f(x) dx$ or $\int \cos f(x) dx$
or $\int \tan f(x) dx$

Example 20

Integrate $\int \sin 3x dx$

Solution

From $\int \sin 3x dx$

Let $u = 3x$, $dx = \frac{du}{3}$

$$\int \sin u \left(\frac{du}{3} \right) \quad \text{thus} \quad \frac{1}{3} \int \sin u du = -\frac{1}{3} \cos u + A \quad \text{But } u = 3x$$

$$\therefore \int \sin 3x dx = -\frac{1}{3} \cos 3x + A$$

Example 21

Find $\int (\cos 5x + \tan 2x) dx$

Solution

$$\int (\cos 5x + \tan 2x) dx \quad \text{thus} \quad \int \cos 5x dx + \int \tan 2x dx = \frac{1}{5} \sin 5x - \frac{1}{2} \ln|\cos 2x| + A$$

Example 22

Integrate $\int \sin 4x \cos 5x dx$

Solution

From $\int \sin 4x \cos 5x dx$

Recall: $\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$

$$\frac{1}{2} \int 2 \sin 4x \cos 5x dx = \frac{1}{2} \int (\sin 9x - \sin x) dx$$

$$\therefore \int \sin 4x \cos 5x dx = -\frac{1}{18} \cos 9x + \frac{1}{2} \cos x + A$$

Example 23

Evaluate the integral $\int \cos^7 x dx$

Solution

$$\int \cos^7 x dx = \int \cos^6 x \cos x dx \text{ thus } \int (\cos^2 x)^3 \cos x dx$$

$$\int (1 - \sin^2 x)^3 \cos x dx$$

Let $u = \sin x$ thus

$$du = \cos x dx \text{ substituting } \int (1 - u^2)^3 du \text{ then } \int (1 - 3u^2 + 3u^4 - u^6) du$$

$$\int (1 - u^2)^3 = u - u^3 + \frac{3}{5} u^5 - \frac{1}{7} u^7 + A$$

But $u = \sin x$

$$\text{Therefore } \int \cos^7 x dx = \sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x + A$$

INTEGRATION BY PART

Refer product rule of differentiation $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

Then $d(uv) = u dv + v du$

Integrating both sides $\int d(uv) = \int u dv + \int v du$ thus $uv = \int u dv + \int v du$

Therefore $\int u dv = uv - \int v du$

This is the formula of integrating by part

To get the function u consider the word ILATE where,

I – Inverse of trigonometric functions

L – Logarithmic functions

A – Algebraic functions

T – Trigonometric functions (common and natural logarithm)

E – Exponential functions

The function that comes first is u any other function is dv .

Example 24

Evaluate $\int x \cos x dx$

Solution

Consider ILATE, therefore, Algebra comes before Trigonometric

We let $u = x \Rightarrow du = dx$ and $dv = \cos x dx$

$$\int dv = \int \cos x dx \Rightarrow v = \sin x$$

$$\int x \cos x dx = x \sin x - \int \sin x dx \text{ thus } \int x \cos x dx = x \sin x + \cos x + A$$

Example 25

Evaluate $\int x^2 e^x dx$

Solution

Let $u = x^2 \Rightarrow du = 2x dx$ and $\int dv = \int e^x dx \Rightarrow v = e^x$

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx \quad \text{But } \int x e^x dx \text{ need by part integration}$$

$$\int x e^x dx = x e^x - \int e^x dx$$

$$\int x e^x dx = x e^x - e^x + A \text{ thus for } \int x^2 e^x dx = x^2 e^x - 2(x e^x - e^x) + A$$

$$\text{Therefore, } \int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + A$$

Example 26

Find $\int e^x \cos x dx$

Solution

Let $u = \cos x \Rightarrow du = -\sin x dx$ and $dv = e^x dx$ thus

$$\int dv = \int e^x dx \Rightarrow v = e^x$$

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Substituting $\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

$$\int e^x \cos x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx$$

$$2 \int e^x \cos x dx = e^x \cos x + e^x \sin x + A$$

$$\therefore \int e^x \cos x dx = \frac{1}{2} e^x (\cos x + \sin x) + A$$

Example 27

Evaluate $\int x^4 \ln x dx$

Solution

From $\int x^4 \ln x dx$, Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$ and $dv = x^4 dx$

$$\int dv = \int x^4 dx \Rightarrow v = \frac{1}{5} x^5 \text{ thus } \int x^4 \ln x dx = \frac{1}{5} x^5 \ln x - \frac{1}{5} \int x^5 \left(\frac{1}{x} \right) dx$$

$$\therefore \int x^4 \ln x dx = \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + A$$

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The tabular method or tic – tac – toe method

So far we saw the process of evaluating integrals by using by part technique, but what will happen when you have the integral like $\int x^4 e^x dx$? This cause a repetition of a by part method. To avoid repetition of by part process in integration, we introduce this method to simplify the repeating of the by part integration process.

Suppose we want to evaluate $\int x^4 e^x dx$ if we use by part method we will repeat the process 4 – times. To deal with this, we introduce the so called *tabular method or tic – tac – toe method*.

Example 28

Integrate $\int x^4 e^x dx$

Solution

From $\int x^4 e^x dx$

Let $u = x^4$ and $dv = e^x dx$

Consider the table below:-

u	sign	dv
x^4	+	e^x
$4x^3$	-	e^x
$12x^2$	+	e^x
$24x$	-	e^x
24	+	e^x
0		e^x

We multiply by following an arrow direction and the appropriate sign to connect each product is shown. Therefore,

$$\int x^4 e^x dx = (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x + A$$

Example 29

Integrate $\int (\ln x) dx$

Solution

$\int \ln x dx$ Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$ and $dv = dx \Rightarrow v = x$

$\int \ln x dx = x \ln x - \int dx$ therefore $\int \ln x dx = x \ln x - x + A$

Example 30

Integrate $\int \tan^{-1}(x) dx$

Solution

$\int \tan^{-1}(x) dx$, Let $u = \tan^{-1} x \Rightarrow du = \frac{dx}{1+x^2}$ and $dv = dx \Rightarrow v = x$

$\int \tan^{-1}(x) dx = x \tan^{-1} x - \int \frac{x dx}{1+x^2}$ therefore,

$\int \tan^{-1}(x) dx = x \tan^{-1}(x) - \frac{1}{2} \ln|1+x^2| + A$

EXERCISE 4

Evaluate the following integrals by using by part method

1. $\int x \sin x dx$
2. $\int x^3 \cos x dx$
3. $\int \sin^{-1} 3x dx$
4. $\int x^4 \sin x dx$
5. $\int \ln x dx$
6. $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$
7. $\int x^3 e^x dx$
8. $\int \sin^2 x dx$
9. $\int x \sin 3x dx$
10. $\int \left(\frac{\sin^{-1} x}{x^2} \right) dx$
11. $\int (e^{2x} \sin x) dx$
12. $\int \left(\frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} \right) dx$
13. $\int (x^2 + 2) \ln x dx$
14. $\int \frac{\ln x}{(x+1)^2} dx$
15. $\int \frac{x^2 \tan^{-1} x}{1+x^2} dx$
16. $\int x^2 e^{3x} dx$
17. $\int x^2 \ln x dx$
18. $\int e^{-x} \cos x dx$
19. $\int x^3 (\ln x)^2 dx$
20. $\int \frac{x^2 \sin^{-1} x}{(1-x^2)^{3/2}} dx$

Integration by partial fractions

Case 1: When the denominator of the rational function contain linear factors, thus;

$$\frac{f(x)}{(ax+b)(cx+d)+\dots+(kx+q)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \dots + \frac{Z}{kx+q} \text{ where } A, B, Z, a, b, c, d, k, q \text{ are constants.}$$

Example 31

Integrate $\int \frac{x+3}{(x-2)(2x+1)} dx$

Solution

From $\int \frac{x+3}{(x-2)(2x+1)} dx$

Decompose the rational function into small rational fractions

$$\frac{x+3}{(x-2)(2x+1)} = \frac{A}{x-2} + \frac{B}{2x+1}$$

Find the values of A and B

$$\frac{x+3}{(x-2)(2x+1)} = \frac{A}{x-2} + \frac{B}{2x+1}$$

By comparison; $x+3 = A(2x+1) + B(x-2)$

When $x=2$ then $A=1$ and when $x=-1/2$ then $B=-1$

$$\frac{x+3}{(x-2)(2x+1)} = \frac{1}{x-2} - \frac{1}{(2x+1)}$$

$$\int \frac{x+3}{(x-2)(2x+1)} dx = \int \frac{1}{x-2} dx - \int \frac{1}{(2x+1)} dx$$

$$\text{Therefore } \int \frac{x+3}{(x-2)(2x+1)} dx = \ln|x-2| - \frac{1}{2} \ln|2x+1| + A$$

Example 32

Evaluate $\int \frac{x-4}{(3x+1)(2x+1)} dx$

Solution

From $\int \frac{x-4}{(3x+1)(2x+1)} dx$

Let $\frac{x-4}{(3x+1)(2x+1)} = \frac{A}{3x+1} + \frac{B}{2x+1}$ then

$$x-4 = A(2x+1) + B(3x+1)$$

Compare the coefficients, $\begin{cases} 2A+3B=1 \\ A+B=-4 \end{cases}$

Solve for A and B , using a calculator, $A = -13$ and $B = 9$

$$\frac{x^2-4x}{(3x+1)(2x+1)} = -\frac{13}{3x+1} + \frac{9}{2x+1}$$

$$\int \frac{x^2-4x}{(3x+1)(2x+1)} dx = -\int \frac{13}{3x+1} dx + \int \frac{9}{2x+1} dx \quad \text{therefore}$$

$$\int \frac{x^2-4x}{(3x+1)(2x+1)} dx = -\frac{13}{3} \ln|3x+1| + \frac{9}{2} \ln|2x+1| + A$$

Example 33

Solve $\int \frac{x+3}{x^2-4} dx$

Solution

From $\int \frac{x+3}{x^2-4} dx$ let $\frac{x+3}{x^2-4} = \frac{x+3}{(x-2)(x+2)}$

Writing into partial fractions as $\frac{x+3}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

Thus $x+3 = A(x+2) + B(x-2)$, when $x=2$ then $A = \frac{5}{4}$ and when

$x=-2$ then $B = -\frac{1}{4}$ thus, $\frac{x+3}{(x-2)(x+2)} = \frac{5}{4(x-2)} - \frac{1}{4(x+2)}$

$$\int \frac{x+3}{x^2-4} dx = \frac{5}{4} \int \frac{1}{x-2} dx - \frac{1}{4} \int \frac{1}{x+2} dx \quad \text{therefore}$$

$$\int \frac{x+3}{x^2-4} dx = \frac{5}{4} \ln|x-2| + \frac{1}{4} \ln|x+2| + A$$

Example 34

Find $\int \frac{x-7}{3x^2-5x-2} dx$

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Solution

Partialize $\frac{x-7}{3x^2-5x-2}$ by factorize the denominator (if possible)

$$3x^2 - 5x - 2 = (3x+1)(x-2)$$

$$\frac{x-7}{3x^2-5x-2} = \frac{x-7}{(3x+1)(x-2)}$$

$$\frac{x-7}{(3x+1)(x-2)} = \frac{A}{3x+1} + \frac{B}{x-2} \text{ thus } x-7 = Ax-2A+3Bx+B$$

$$\text{Compare coefficients and solve for } A \text{ and } B \begin{cases} A+3B=1 \\ -2A+B=-7 \end{cases}$$

$$A = \frac{22}{7} \text{ and } B = -\frac{5}{7} \text{ thus } \frac{x-7}{(3x+1)(x-2)} = \frac{22}{7(3x+1)} - \frac{5}{7(x-2)}$$

$$\int \frac{x-7}{3x^2-5x-2} dx = \frac{22}{7} \int \frac{1}{3x+1} dx - \frac{5}{7} \int \frac{1}{x-2} dx$$

$$\therefore \int \frac{x-7}{3x^2-5x-2} dx = \frac{22}{21} \ln|3x+1| - \frac{5}{7} \ln|x-2| + A$$

EXERCISE 5

By partial fraction, integrate the following

1. $\int \frac{2x-1}{(x-2)(3x+2)} dx$

2. $\int \frac{x}{(x+1)(x+2)} dx$

3. $\int \frac{x+2}{(x-2)(3-2x)} dx$

4. $\int \frac{dx}{8-2x^2}$

5. $\int \frac{x^2}{(2+x^3)(1-x^3)} dx$

6. $\int \frac{\cos x}{(1-\sin x)(2-\sin x)} dx$

7. $\int \frac{dx}{x(x^4-1)} = \int \frac{x^3 dx}{x^4(x^4-1)}$

8. $\int \frac{x}{(x-2)(x+1)(x-3)} dx$

9. $\int \frac{dx}{x(x-1)(x+1)}$

10. $\int \frac{1}{\sin x + \sin 2x} dx$

Case 2: When the denominator contain quadratic factor(s)

$$\frac{f(x)}{(ax^2+b)(cx^2+d)\dots+(px+q)} = \frac{Ax+B}{ax^2+b} + \frac{Cx+D}{cx^2+d} + \dots + \frac{E}{px+q} \quad \text{where}$$

$A, B, C, D, E, a, b, c, d, p, q$ are constants.

Example 35

Evaluate $\int \frac{x+2}{(x-3)(x^2+2)} dx$

Solution

Partialize $\frac{x+2}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$

$$x+2 = A(x^2+2) + (Bx+C)(x-3)$$

$$x+2 = Ax^2 + 2A + Bx^2 - 3Bx + Cx - 3C$$

Compare the coefficients
$$\begin{cases} -3B + C = 1 \\ 2A - 3C = 2 \\ A + B = 0 \end{cases}$$

Then, $A = \frac{5}{11}$, $B = -\frac{5}{11}$ and $C = -\frac{4}{11}$ thus

$$\frac{x+2}{(x-3)(x^2+2)} = \frac{\frac{5}{11}}{x-3} - \frac{\frac{5}{11}x + \frac{4}{11}}{x^2+2}$$

Therefore $\int \frac{x+2}{(x-3)(x^2+2)} dx = \frac{5}{11} \int \frac{1}{x-3} dx - \frac{1}{11} \int \frac{5x+4}{x^2+2} dx$

$$\int \frac{x+2}{(x-3)(x^2+2)} dx = \frac{5}{11} \ln \left| \frac{x-3}{\sqrt{x^2+2}} \right| - \frac{2\sqrt{2}}{11} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + A$$

Example 36

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Evaluate $\int \frac{x+3}{x(x^2+1)} dx$

Solution

$$\text{Partialize } \frac{x^2+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x^2+3 = A(x^2+1) + x(Bx+C)$$

$$x^2+3 = Ax^2 + A + Bx^2 + Cx \text{ then } \begin{cases} A+B=1 \\ A=3 \\ B=-2 \\ C=0 \end{cases}$$

$$\int \frac{x^2+3}{x(x^2+1)} dx = \int \frac{3}{x} dx - \int \frac{2x}{x^2+1} dx$$

$$\int \frac{x^2+3}{x(x^2+1)} dx = 3\ln|x| - \ln|x^2+1| + A$$

$$\int \frac{x^2+3}{x(x^2+1)} dx = \ln \left| \frac{x^3}{x^2+1} \right| + A$$

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Example 37

Evaluate $\int \frac{x^2-4x+3}{2x^3-x^2+18x-9} dx$

Solution

$$\frac{x^2-4x+3}{2x^3-x^2+18x-9} = \frac{x^2-4x+3}{(x^2+9)(2x-1)}$$

$$\text{Partialize } \frac{x^2-4x+3}{(x^2+9)(2x-1)} = \frac{Ax+B}{x^2+9} + \frac{C}{2x-1}$$

$$x^2-4x+3 = (Ax+B)(2x-1) + C(x^2+9)$$

$$x^2-4x+3 = 2Ax^2 - Ax + 2Bx - B + Cx^2 + 9C$$

Compare the coefficients

$$\begin{cases} 2A + C = 1 \\ -A + 2B = -4 \\ -B + 9C = 3 \end{cases}$$

Then, $A = \frac{16}{37}$, $B = -\frac{66}{37}$ and $C = \frac{5}{37}$

$$\begin{aligned} & \frac{1}{37} \int \frac{16x - 66}{x^2 + 9} dx + \frac{5}{37} \int \frac{1}{2x - 1} dx \\ & \frac{16}{37} \int \frac{x}{x^2 + 9} dx - \frac{66}{37} \int \frac{1}{x^2 + 9} dx + \frac{5}{37} \int \frac{1}{2x - 1} dx \\ & = \frac{8}{37} \ln|x^2 + 9| - \frac{22}{37} \tan^{-1}\left(\frac{x}{3}\right) + \frac{5}{74} \ln|2x - 1| + A \end{aligned}$$

EXERCISE 6

Evaluate the following integrals

1. $\int \frac{x}{(x+1)(x^2+3)} dx$

3. $\int \left(\frac{x}{x^3 - x^2 + x - 1} \right) dx$

5. $\int \frac{x-3}{(x^2+1)(x-4)} dx$

7. $\int \frac{x+5}{(x^2+2)(x-2)} dx$

9. $\int \frac{dx}{(1+2x^2)(x-4)}$

2. $\int \frac{dx}{x(x^6+1)}$

4. $\int \frac{dx}{x^4-1}$

6. $\int \frac{1}{x(x^2+1)} dx$

8. $\int \frac{x}{(x^2+1)(3x-1)} dx$

10. $\int \frac{dx}{x(x^2+2)}$

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Case 3: When the denominator contains the repeated root(s).

$$\frac{f(x)}{(ax+b)^n \dots (px+q)} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^{n-1}} + \dots + \frac{C}{(ax+b)^{n-r}} + \frac{D}{px+q}$$

where A, B, C, D, a, b, p, q are constants and n is a positive integer, where r is less than or equal to n .

Example 38

Evaluate $\int \frac{x+1}{3x(x+2)^2} dx$

Solution

Partialize $\frac{x+1}{3x(x+2)^2} = \frac{A}{3x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$$\frac{x+1}{3x(x+2)^2} = \frac{A}{3x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$x+1 = A(x+2)^2 + 3Bx(x+2) + 3Cx$$

When $x=0$ then $A=1/4$, when $x=-2$ then $C=1/6$ and when $x=1$ then $B=-1/12$

$$\frac{x+1}{3x(x+2)^2} = \frac{1}{12x} - \frac{1}{12(x+2)} + \frac{1}{6(x+2)^2}$$

$$\int \frac{x+1}{3x(x+2)^2} dx = \frac{1}{12} \int \frac{1}{x} dx - \frac{1}{12} \int \frac{1}{x+2} dx + \frac{1}{6} \int \frac{1}{(x+2)^2} dx$$

Therefore, $\int \frac{x+1}{3x(x+2)^2} dx = \frac{1}{12} \ln|x| - \frac{1}{12} \ln|x+2| - \frac{1}{6(x+2)} + A$

Example 39

Evaluate $\int \frac{3x-7}{(x+5)(2x-1)^2} dx$

Solution

Partialize $\frac{3x-7}{(x+5)(2x-1)^2} = \frac{A}{x+5} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$

$$\frac{3x-7}{(x+5)(2x-1)^2} = \frac{A}{x+5} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$$

$$3x-7 = A(2x-1)^2 + B(x+5)(2x-1) + C(x+5)$$

$$\begin{cases} A-5B+5C=-7 & \text{when } x=0 \\ A+6B+6C=-4 & \text{when } x=1 \\ 9A-12B+4C=-10 & \text{when } x=-1 \end{cases} \quad \text{thus } A=-\frac{2}{11}, B=\frac{4}{11} \text{ and } C=-1$$

$$\frac{3x-7}{(x+5)(2x-1)^2} = -\frac{2}{11(x+5)} + \frac{4}{11(2x-1)} - \frac{1}{(2x-1)^2}$$

$$\text{Then } -\frac{2}{11} \int \frac{1}{(x+5)} dx + \frac{4}{11} \int \frac{1}{(2x-1)} dx - \int \frac{1}{(2x-1)^2} dx$$

$$= -\frac{2}{11} \ln|x+5| + \frac{2}{11} \ln|2x-1| + \frac{1}{2} \left(\frac{1}{2x-1} \right) + A$$

$$\text{Therefore } \int \frac{3x-7}{(x+5)(2x-1)^2} = \frac{2}{11} \ln \left| \frac{2x-1}{x+5} \right| + \frac{1}{4x-2} + A$$

EXERCISE 7

Evaluate the following integrals

1. $\int \frac{dx}{x^2(x+1)}$

2. $\int \frac{dx}{x(x+1)(x+2)}$

3. $\int \frac{x+2}{(x+1)(x+3)^2} dx$

4. $\int \frac{x}{(x+1)^2(2x-1)} dx$

5. $\int \frac{x+3}{(x-2)(x-1)^3} dx$

6. $\int \frac{x^2+x+3}{(2x+3)(3x-1)^2} dx$

7. $\int \frac{3x-1}{(x-2)^3} dx$

8. $\int \frac{x^2-3x+1}{(3x-5)^2(x^2+1)} dx$

9. $\int \frac{4x}{(x-1)^2(x+1)^2} dx$

10. $\int \frac{x+4}{(2x^2-1)(x+3)(x-1)} dx$

Case 4: When the denominator has no real factors or cannot be factorized easily

$$\frac{f(x)}{ax^n + bx^{n-2} + cx^{n-r} + dx^{n-k}}$$

Example 40

Find (a) $\int \frac{4}{x^2+2x+2} dx$

(b) $\int \frac{dx}{2x^2+11x+3}$

Solution

(a) $\int \frac{4}{x^2 + 2x + 2} dx$

Consider $x^2 + 2x + 2 = (x+1)^2 + 1$ therefore, $\frac{4}{x^2 + 2x + 2} = \frac{4}{(x+1)^2 + 1}$

$$\int \frac{4}{x^2 + 2x + 2} dx = \int \frac{4}{(x+1)^2 + 1} dx \quad \text{thus} \quad \int \frac{4}{(x+1)^2 + 1} dx = 4 \int \frac{1}{(x+1)^2 + 1} dx$$

Let $x+1 = \tan \theta$

Hence $dx = \sec^2 \theta d\theta$ thus $4 \int \frac{\sec^2 \theta d\theta}{\tan^2 \theta + 1} = 4\theta + A$

Therefore, $\int \frac{4}{x^2 + 2x + 2} dx = 4 \tan^{-1}(x+1) + A$

(b) $\int \frac{dx}{2x^2 + 11x + 3}$

The expression $2x^2 + 11x + 3 = 2\left(x + \frac{11}{4}\right)^2 - \frac{97}{8}$

therefore, $\int \frac{dx}{2x^2 + 11x + 3} = \int \frac{dx}{2\left(x + \frac{11}{4}\right)^2 - \frac{97}{8}}$

$$= -\frac{8}{97} \int \frac{dx}{1 - \frac{16}{97}\left(x + \frac{11}{4}\right)^2} = -\frac{8}{97} \int \frac{dx}{1 - \left(\frac{4}{\sqrt{97}}\left(x + \frac{11}{4}\right)\right)^2}$$

(See the hyperbolic substitution section)

Let $\tanh \theta = \frac{4}{\sqrt{97}}\left(x + \frac{11}{4}\right)$ then $\frac{\sqrt{97}}{4} \tanh \theta = x + \frac{11}{4}$ hence

$$\frac{\sqrt{97}}{4} \operatorname{sech}^2 \theta d\theta = dx$$

Substituting $-\frac{8}{97} \int \frac{\frac{\sqrt{97}}{4} \operatorname{sech}^2 \theta d\theta}{1 - \tanh^2 \theta} = -\frac{8}{97} \times \frac{\sqrt{97}}{4} \int d\theta$

$$= -\frac{2}{\sqrt{97}} \theta + C \quad \text{replacing } \theta, \text{ then } -\frac{2}{\sqrt{97}} \tanh^{-1}\left(\frac{4x+11}{\sqrt{97}}\right) + C$$

Therefore, $\int \frac{dx}{2x^2 + 11x + 3} = -\frac{2}{\sqrt{97}} \tanh^{-1} \left(\frac{4x+11}{\sqrt{97}} \right) + A$

EXERCISE 8

Evaluate the following

1. $\int \frac{dx}{2x^2 - x + 3}$

2. $\int \frac{dx}{5x^2 + 7x + 1}$

3. $\int \frac{dx}{2x^2 + 8}$

4. $\int \frac{dx}{2x^2 + 2x + 1}$

5. $\int \frac{dx}{\sqrt{x^2 - 2x + 1}}$

6. $\int \frac{dx}{\sqrt{x^2 - 2x + 5}}$

7. $\int \frac{dx}{2x^2 + 4x + 11}$

8. $\int \frac{dx}{4x^2 - 8x + 7}$

9. $\int \frac{2}{x^2 + 3x - 2} dx$

10. $\int \frac{5}{3x^2 - 3x + 2} dx$

Case 5: The use of $N = A \frac{d}{dx}(D) + B$, where N is numerator, D is denominator, A and B are constants, $\frac{f(x)}{g(x)}$ then $f(x) = A \frac{dg}{dx} + B$

Example 41

Evaluate $\int \frac{2x-3}{x^2 + 3x + 5} dx$

Solution

From $\frac{2x-3}{x^2 + 3x + 5}$ Let $N = A \frac{d}{dx}(D) + B$ then $2x-3 = A(2x+3) + B$

$$2x-3 = 2Ax + 3A + B \text{ hence } A=1 \text{ and } B=-6$$

$$2x-3 = 2x + 3 - 6$$

$$\frac{2x-3}{x^2 + 3x + 5} = \frac{2x + 3 - 6}{x^2 + 3x + 5}$$

$$\begin{aligned}\frac{2x+3-6}{x^2+3x+5} &= \frac{2x+3}{x^2+3x+5} - \frac{6}{x^2+3x+5} \\ \int \frac{2x+3-6}{x^2+3x+5} dx &= \int \frac{2x+3}{x^2+3x+5} - \int \frac{6}{x^2+3x+5} dx \\ \int \frac{2x-3}{x^2+3x+5} dx &= \ln|x^2+3x+5| - \frac{12}{\sqrt{11}} \tan^{-1}\left(\frac{2x+3}{\sqrt{11}}\right) + A\end{aligned}$$

Example 42

Evaluate $\int \frac{x^2+3x-3}{x^2-6x+8} dx$

Solution

The rational function involved in this integral is an improper rational function. So divide first to lower the numerator degree.

Use long division method, $x^2 - 6x + 8 \overline{) x^2 + 3x - 3}$

$$\begin{array}{r} x^2 - 6x + 8 \\ 9x - 11 \end{array}$$

$$1 + \frac{9x-11}{x^2-6x+8}$$

$$\int dx + \int \frac{9x-11}{x^2-6x+8} dx$$

Now Partialize,

$$\frac{9x-11}{x^2-6x+8} = \frac{9x-11}{(x-2)(x-4)}$$

$$\frac{9x-11}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}$$

$$9x-11 = A(x-4) + B(x-2)$$

When $x=4$, $B = \frac{25}{2}$ and when $x=2$, $A = -\frac{7}{2}$

$$\int dx + \frac{25}{2} \int \frac{dx}{x-3} - \frac{7}{2} \int \frac{dx}{x-2}$$

$$\text{Therefore, } \int \frac{x^2+3x-3}{x^2-6x+8} dx = x + \frac{25}{2} \ln(x-4) - \frac{7}{2} \ln(x-2) + A$$

Example 43

$$\text{Evaluate } \int \frac{x^4+2}{x^2+5} dx$$

Solution

$$\text{From } \int \frac{x^4+5}{x^2+2} dx$$

$$\text{Dividing by long division } x^2+2 \overline{) \begin{array}{r} x^2-2 \\ x^4+0x^2+5 \\ -x^4+2x^2 \\ \hline -2x^2+5 \end{array}}$$

$$\begin{array}{r} -2x^2+5 \\ -2x^2-4 \\ \hline 9 \end{array}$$

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$$\frac{x^4+5}{x^2+2} = x^2 - 2 + \frac{9}{x^2+2}$$

$$\int \frac{x^4+5}{x^2+2} dx = \int (x^2-2) dx + \int \frac{9}{x^2+2} dx$$

$$\int \frac{x^4+5}{x^2+2} dx = \frac{x^3}{3} - 2x + \frac{9}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + A$$

EXERCISE 9

Evaluate the following integrals

$$1. \int \frac{2x-1}{2x^2+3x+2} dx$$

$$2. \int \frac{x+4}{x^2+x+1} dx$$

3. $\int \frac{3x+2}{x^2+2x+2} dx$

4. $\int \frac{2x+5}{x^2+4x-7} dx$

5. $\int \frac{2x+5}{x^2+4} dx$

6. $\int \frac{2x+3}{2x^2+6} dx$

7. $\int \frac{x+2}{x(x^2+4)} dx$

8. $\int \frac{x+3}{2x^2-x+1} dx$

9. $\int \frac{5-3x}{x^2+3x+4} dx$

10. $\int \frac{x+1}{2x^2-x-3} dx$

11. $\int \frac{x-1}{x^2-x+3} dx$

12. $\int \frac{2-x}{3x^2+9x+1} dx$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Case 6: The use of $N = \frac{d}{dx}(D) + BD$ (for trigonometric) where N is Numerator, D is denominator and A and B are constants.

Integrals of the form $\int \frac{c_1 \sin b_1 x + c_2 \cos b_1 x}{c_3 \sin b_3 x + c_4 \cos b_3 x} dx$

Example 44

Evaluate $\int \frac{\sin x}{\cos x - 2 \sin x} dx$

Solution

From $\int \frac{\sin x}{\cos x - 2 \sin x} dx$ Let $N = \frac{d}{dx}(D) + BD$

$$\sin x = A \frac{d}{dx}(\cos x - 2 \sin x) + B(\cos x - 2 \sin x)$$

$$\sin x = -A \sin x - 2A \cos x + B \cos x - 2B \sin x$$

$$\text{Compare } \begin{cases} -A - 2B = 1 \\ -2A + B = 0 \end{cases}$$

Solve for A and B , $A = -\frac{1}{5}$ and $B = -\frac{2}{5}$

$$\int \frac{\sin x}{\cos x - 2 \sin x} dx = \frac{1}{5} \int \frac{(\sin x + 2 \cos x)}{\cos x - 2 \sin x} dx - \frac{2}{5} \int \frac{(\cos x - 2 \sin x)}{\cos x - 2 \sin x} dx$$

Consider

$$\frac{1}{5} \int \frac{\sin x + 2 \cos x}{\cos x - 2 \sin x} dx, \quad \text{Let } u = \cos x - 2 \sin x$$

$$du = -(\sin x + 2 \cos x) dx$$

$$\frac{1}{5} \int \frac{du}{u} = -\frac{1}{5} \ln|u|, \quad \text{But } u = \cos x - 2 \sin x$$

$$\frac{1}{5} \int \frac{\sin x + 2 \cos x}{\cos x - 2 \sin x} dx = -\frac{1}{5} \ln|\cos x - 2 \sin x|$$

Also consider

$$\frac{2}{5} \int \frac{\cos x - 2 \sin x}{\cos x - 2 \sin x} dx = -\frac{2}{5} \int dx = -\frac{2}{5} x$$

$$\text{Therefore, } \int \frac{\sin x}{\cos x - 2 \sin x} dx = -\frac{1}{5} \ln|\cos x - 2 \sin x| - \frac{2}{5} x + A$$

Example 45

$$\text{Evaluate } \int \frac{\cos x}{3 \cos x + \sin x} dx$$

Solution

$$\text{From } \int \frac{\cos x}{3 \cos x + \sin x} dx$$

$$\cos x = A \frac{d}{dx} (3 \cos x + \sin x) + B(3 \cos x + \sin x)$$

$$\cos x = A \cos x - 3A \sin x + 3B \cos x + B \sin x$$

$$\text{Compare coefficients } \begin{cases} A + 3B = 1 \\ -3A + B = 0 \end{cases}$$

$$\text{Solve for } A \text{ and } B \text{ then } A = \frac{1}{10} \text{ and } B = \frac{3}{10}$$

$$\int \frac{\cos x}{3 \cos x + \sin x} dx = \frac{1}{10} \int \frac{\cos x - 3 \sin x}{3 \cos x + \sin x} dx + \frac{3}{10} \int \frac{3 \cos x + \sin x}{3 \cos x + \sin x} dx$$

$$\int \frac{\cos x}{3 \cos x + \sin x} dx = \frac{1}{10} \ln|3 \cos x + \sin x| + \frac{3}{10} x + A$$

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EXERCISE 10

Evaluate the following integrals

1. $\int \frac{\sin 2x}{\cos 2x + \sin 2x} dx$

2. $\int \frac{2 \sin x + \cos x}{7 \sin x - 5 \cos x} dx$

3. $\int \frac{2 \sin x + 3 \cos x}{3 \sin x + 4 \cos x} dx$

4. $\int \frac{3 \cos x}{\sqrt{3 \sin x + 4 \cos x}} dx$

5. $\int \frac{5 \sin x}{\sqrt{\cos x}} dx$

6. $\int \frac{4 \sin x}{\sqrt{2 \cos x + 5}} dx$

7. $\int \frac{\sin x}{\cos x + \sin x} dx$

8. $\int \frac{3 \cos x}{\cos x + 4 \sin x} dx$

9. $\int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

10. $\int \frac{\tan x}{\cos x} dx$

Integrals of the form

(a) $\int \frac{dx}{a + b \cos x}$

(b) $\int \frac{dx}{a + b \sin x}$

(c) $\int \frac{dx}{a \cos x + b \sin x}$

Use of t formulaRefer, $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$ where $t = \tan\left(\frac{x}{2}\right)$

Example 46

Solve $\int \frac{dx}{3 + 5 \cos x}$

Solution

$$\int \frac{dx}{3 + 5 \cos x} \quad \text{Let } t = \tan\left(\frac{x}{2}\right) \quad \text{then } \frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{x}{2}\right)$$

$$\text{Thus } \frac{2dt}{\sec^2\left(\frac{x}{2}\right)} = dx$$

But $\cos x = \frac{1-t^2}{1+t^2}$

$$\int \frac{2dt}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \text{ then } \int \frac{2dt}{3(1+t^2)+5(1-t^2)} = \int \frac{2dt}{8-2t^2}$$

$$\int \frac{dt}{4-t^2} = \frac{1}{4} \int \frac{dt}{1-\left(\frac{t}{2}\right)^2}, \quad \text{Let } \frac{t}{2} = \tanh \theta \text{ thus } dt = 2\operatorname{sech}^2 \theta d\theta$$

$$\frac{1}{4} \int \frac{2\operatorname{sech}^2 \theta}{1-\tanh^2 \theta} d\theta = \frac{1}{2} \theta + A$$

But $\theta = \tanh^{-1}\left(\frac{t}{2}\right)$ and $t = \tan\left(\frac{x}{2}\right)$

$$\int \frac{dx}{3+5\cos x} = \tanh^{-1}\left(\frac{1}{2} \tan\left(\frac{x}{2}\right)\right) + A$$

Example 47

Solve $\int \frac{dx}{5-3\cos x}$

Solution

From $\int \frac{dx}{5-3\cos x}$ Let $t = \tan\left(\frac{x}{2}\right)$ then $dx = \frac{2dt}{\sec^2\left(\frac{x}{2}\right)}$ therefore

$$dx = \frac{2dt}{1+t^2}$$

$$\int \frac{2dt}{5-3\left(\frac{1-t^2}{1+t^2}\right)} = \int \frac{2dt}{2+8t^2} \text{ thus } \int \frac{2dt}{2+8t^2} = \int \frac{dt}{1+(2t)^2}$$

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$$2t = \tan \theta \Rightarrow dt = \frac{1}{2} \sec^2 \theta d\theta \text{ therefore}$$

$$\int \frac{dt}{1+(2t)^2} = \frac{1}{2} \int \frac{\sec^2 \theta d\theta}{1+\tan^2 \theta} = \frac{1}{2} \theta + A \quad \text{But } \theta = \tan^{-1}(2t) \text{ and}$$

$$t = \tan\left(\frac{x}{2}\right) \text{ thus } \int \frac{dx}{5-3\cos x} = \frac{1}{2} \tan^{-1}\left(2 \tan\left(\frac{x}{2}\right)\right) + A$$

Example 48

Integrate $\int \frac{dx}{1+\sin x}$

Solution

From $\int \frac{dx}{1+\sin x}$, Let $t = \tan\left(\frac{x}{2}\right)$ then $\frac{dt}{dx} = \frac{1}{2} \sec^2\left(\frac{x}{2}\right)$ thus

$$dx = \frac{2dt}{\sec^2\left(\frac{x}{2}\right)} = \frac{2dt}{1+t^2} \text{ substituting } \int \frac{\frac{2}{1+t^2}}{1+\frac{2t}{1+t^2}} dt = \int \frac{2}{1+2t+t^2} dt$$

$$\int \frac{2}{1+2t+t^2} dt = \int \frac{2}{(t+1)^2} dt \quad \text{Let } u = t+1 \text{ then } du = dt$$

$$\int \frac{2du}{u^2} = -\frac{2}{u} + A \quad \text{but } u = t+1$$

$$\int \frac{2dt}{(t+1)^2} = -\frac{2}{t+1} + A \quad \text{but } t = \tan\left(\frac{x}{2}\right)$$

$$\int \frac{dx}{1+\sin x} = -\frac{2}{\tan\left(\frac{x}{2}\right)+1} + A$$

Example 49

Evaluate $\int \frac{dx}{3\cos x + 4\sin x + 6}$

Solution

From $\int \frac{dx}{3 \cos x + 4 \sin x + 6},$

Let $t = \tan\left(\frac{x}{2}\right) \Rightarrow dt = \frac{1}{2} \sec^2\left(\frac{x}{2}\right) dx$ then

$$\int \frac{\frac{2dt}{1 + \tan^2\left(\frac{x}{2}\right)}}{3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) + 6}$$

$$\int \frac{\frac{2dt}{1+t^2}}{3-3t^2+8t+6+6t^2}$$

$$\frac{2}{3} \int \frac{dt}{t^2 + \frac{8}{3}t + 3}$$

$$\frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{3t+4}{\sqrt{11}}\right) + A$$

$$\frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{3 \tan\left(\frac{x}{2}\right) + 4}{\sqrt{11}}\right) + A$$

$$\int \frac{2dt}{3t^2 + 8t + 9}$$

$$\frac{2}{3} \int \frac{dt}{\left(t + \frac{4}{3}\right)^2 + \frac{11}{9}}$$

But $t = \tan\left(\frac{x}{2}\right)$

Example 50

Evaluate $\int \frac{dx}{(2 \sin x + \cos x)^2}$

Solution

From $\int \frac{dx}{(2 \sin x + \cos x)^2}$ Divide by $\cos x$

$$\int \frac{\sec x dx}{(2 \tan x + 1)^2} \text{ thus, } -\frac{1}{2(2 \tan x + 1)}$$

EXERCISE 11

Evaluate the following

1. $\int \frac{dx}{3 \cos x - 4 \sin x + 5}$

3. $\int \frac{dx}{1 - \sin x \cos x}$

5. $\int \frac{dx}{2 \sin x + \cos x + 3}$

7. $\int \frac{\sin x}{2 \cos x + 3 \sin x} dx$

9. $\int \frac{\cos x}{1 + \cos x + \sin x} dx$

2. $\int \frac{dx}{4 + 5 \sin x}$

4. $\int \frac{dx}{12 \cos x - 5 \sin x + 13}$

6. $\int \frac{dx}{2 + 3 \tan x}$

8. $\int \frac{\sin x + \cos x}{2 \sin x + \cos x} dx$

10. $\int \frac{1}{2 + \sin 2x} dx$

R- FORMULA IN INTEGRATION

$$r \cos(\theta \pm \alpha) = r \cos \theta \cos \alpha \mp r \sin \theta \sin \alpha \text{ or}$$

$$r \sin(\theta \pm \alpha) = r \sin \theta \cos \alpha \pm r \cos \theta \sin \alpha$$

Example 51

Evaluate $\int \frac{dx}{3 \cos x + 4 \sin x}$

Solution

$$\int \frac{dx}{3 \cos x + 4 \sin x} \quad \text{Let } 3 \cos x + 4 \sin x = r \sin(x + \alpha)$$

$$3 \cos x + 4 \sin x = r \sin x \cos \alpha + r \cos x \sin \alpha$$

$$r \cos x \sin \alpha = 3 \cos x$$

$$r \sin \alpha = 3 \quad (1)$$

$$r \sin x \cos \alpha = 4 \sin x$$

$$r \cos \alpha = 4 \quad (2)$$

Divide equation (1) by equation (2)

$$\alpha = \tan^{-1} \left(\frac{3}{4} \right)$$

Square equation (1) and (2) and sum then

$$r = \sqrt{3^2 + 4^2} = 5$$

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$$3 \cos x + 4 \sin x = 5 \sin \left(x + \tan^{-1} \left(\frac{3}{4} \right) \right)$$

$$\int \frac{dx}{5 \sin \left(x + \tan^{-1} \left(\frac{3}{4} \right) \right)}$$

$$\frac{1}{5} \int \operatorname{cosec} \left(x + \tan^{-1} \left(\frac{3}{4} \right) \right) dx$$

$$\frac{1}{5} \ln \left| \operatorname{cosec} \left(x + \tan^{-1} \left(\frac{3}{4} \right) \right) - \cot \left(x + \tan^{-1} \left(\frac{3}{4} \right) \right) \right| + A$$

EXERCISE 12

Evaluate the following

1. $\int \frac{dx}{5 \cos x - 12 \sin x}$

3. $\int \frac{dx}{24 \cos x + 7 \sin x}$

5. $\int \frac{\sec^2 x dx}{\tan x + \sqrt{3}}$

7. $\int \frac{\sin x + 2 \cos x}{\cos x - \sin x} dx$

9. $\int \frac{1}{\sin 2x + 3 \cos 2x} dx$

2. $\int \frac{dx}{\cos x + \sin x}$

4. $\int \frac{dx}{\sqrt{2}(\sin x - \cos x)}$

6. $\int \frac{\cos x}{\cot x + 4} dx$

8. $\int \frac{4}{3 + \sin x} dx$

10. $\int \frac{\sec x}{\tan x + 4} dx$

Integral of the form $\int \cos^m x \sin^n x dx$

Example 52

Evaluate $\int \sin^2 x dx$

Solution

Given $\int \sin^2 x dx$ from $\cos 2x = \cos^2 x - \sin^2 x$ thus

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \sin^2 x dx = \frac{1}{2} \int (1 - \cos 2x) dx \text{ therefore } \int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + A$$

Example 53

Integrate $\int \cos^3 x \sin x dx$

Solution

$$\text{Let } u = \cos x \Rightarrow dx = -\frac{du}{\sin x}$$

$$-\int u^3 \sin x \frac{du}{\sin x} = -\int u^3 du = -\frac{1}{4}u^4 + A \quad \text{But } u = \cos x \text{ then}$$

$$\int \cos^3 x \sin x dx = -\frac{1}{4}\cos^4 x + A$$

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Example 54

Evaluate $\int \cos^3 x \sin^2 x dx$

Solution

$$\int \cos^3 x \sin^2 x dx = \int \cos x \cos^2 x \sin^2 x dx$$

$$\int \cos x \cos^2 x \sin^2 x dx = \int \cos x (1 - \sin^2 x) \sin^2 x dx$$

$$\text{Let } u = \sin x \Rightarrow du = \cos x dx$$

$$\int (1 - u^2) u^2 du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + A \quad \text{But } u = \sin x$$

$$\int \cos^3 x \sin^2 x dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + A$$

EXERCISE 13

Evaluate the following integrals

1. $\int \sin^5 x \cos^2 x dx$

2. $\int \sin^3 x \cos^4 x dx$

3. $\int \sin^7 x dx$

4. $\int \sin^8 x \cos^5 x dx$

5. Show that $\int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + A$

6. $\int \cos^5 x \sin^7 x dx$

7. $\int \cos^{11} x dx$

8. $\int \sec^4 x \tan x dx$

9. $\int \cos^7 x \sin^2 x dx$

10. Show that $\int_0^{\frac{\pi}{2}} x \sin^2 x dx = \frac{\pi^2}{16} + \frac{1}{4}$

11. $\int \sin^3 x \cos^7 x dx$

12. $\int \cos^4 x dx$

Integrals of the form $\int \frac{1}{a \cos^2 x + b \sin^2 x} dx$

Example 55

Integrate $\int \frac{dx}{3 \cos^2 x + 5 \sin^2 x}$

Solution

From $\int \frac{dx}{3 \cos^2 x + 5 \sin^2 x}$ Divide throughout by $\cos^2 x$

$$\int \frac{\sec^2 x}{3 + 5 \tan^2 x} dx \quad \text{Let } u = \tan x \Rightarrow du = \sec^2 x dx$$

$$\int \frac{du}{3 + 5u^2} = \frac{1}{3} \int \frac{du}{1 + \frac{5}{3}u^2} \quad \text{Let } \frac{\sqrt{5}}{\sqrt{3}}u = \tan \theta$$

$$\frac{\sqrt{5}}{\sqrt{3}} du = \sec^2 \theta d\theta \text{ therefore } du = \frac{\sqrt{3}}{\sqrt{5}} \sec^2 \theta d\theta$$

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$$= \frac{1}{3} \int \frac{\frac{\sqrt{3}}{\sqrt{5}} \sec^2 \theta d\theta}{1 + \tan^2 \theta} = \frac{\sqrt{3}}{3\sqrt{5}} \int d\theta$$

But $\theta = \tan^{-1} \left(\frac{\sqrt{5}}{\sqrt{3}} u \right)$

$$\frac{\sqrt{3}}{3\sqrt{5}} \tan^{-1} \left(\frac{\sqrt{5}}{\sqrt{3}} u \right) + A$$

But $u = \tan x$

$$\frac{\sqrt{15}}{15} \tan^{-1} \left(\frac{\sqrt{15}}{3} \tan x \right) + A$$

Therefore, $\int \frac{dx}{3\cos^2 x + 5\sin^2 x} = \frac{\sqrt{15}}{15} \tan^{-1} \left(\frac{\sqrt{15}}{3} \tan x \right) + A$

Example 56

Solve $\int \frac{dx}{1 - \cos 2x}$

Solution

From $\int \frac{dx}{1 - \cos 2x}$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\int \frac{dx}{2\sin^2 x} = \frac{1}{2} \int \frac{dx}{\sin^2 x} = \frac{1}{2} \int \operatorname{cosec}^2 x dx$$

$$\int \frac{dx}{1 - \cos^2 x} = \frac{1}{2} \int \operatorname{cosec}^2 x dx = -\frac{1}{2} \cot^2 x + A$$

$$\int \frac{dx}{1 - \cos^2 x} = -\frac{1}{2} \cot^2 x + A$$

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Example 57

Evaluate $\int \frac{\cos x}{4 + \sin^2 x} dx$

Solution

From $\int \frac{\cos x}{4 + \sin^2 x} dx$

Let $u = \sin x$ thus $du = \cos x dx$

$$\int \frac{du}{4+u^2} = \frac{1}{4} \int \frac{du}{1+\left(\frac{u}{2}\right)^2}$$

$$\int \frac{du}{4+u^2} = \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) = \frac{1}{2} \tan^{-1}\left(\frac{1}{2} \sin x\right) + A$$

Therefore, $\int \frac{\cos x}{4+\sin^2 x} dx = \frac{1}{2} \tan^{-1}\left(\frac{1}{2} \sin x\right) + A$

Example 58

Evaluate $\int \frac{\sin x}{1+\cos^2 x} dx$

Solution

From $\int \frac{\sin x}{1+\cos^2 x} dx$

$$\int \frac{\sin x}{2\cos^2 x + \sin^2 x} dx = \int \frac{\sec x \tan x}{2 + \tan^2 x} dx$$

$$\int \frac{\sec x \tan x}{1 + \sec^2 x} dx$$

Let $u = \sec x \Rightarrow du = \sec x \tan x dx$

$$\int \frac{du}{1+u^2} = \tan^{-1}(u) + A$$

But $u = \sec x$

$$\int \frac{\sin x}{1+\cos^2 x} dx = \tan^{-1}(\sec x) + A$$

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EXERCISE 14

Evaluate the following

1. $\int \frac{dx}{\cos^2 x + 4 \sin^2 x}$

3. $\int \frac{1}{9 \cos^2 x + \sin^2 x} dx$

5. $\int \frac{1}{\sin^2 x - \sqrt{3} \cos x} dx$

7. $\int \frac{\tan x}{1 + \sin x \cos x} dx$

9. $\int \frac{2}{4 \cos x - \sec x} dx$

2. $\int \frac{dx}{\cos 2x + 3}$

4. $\int \frac{1}{2 + \cos x \sin x} dx$

6. $\int \frac{\sin x}{\cos x - \sin x} dx$

8. $\int \frac{1}{2 + \cos 2x} dx$

10. $\int \frac{\cos x + \sin x}{\cos x - \sin x} dx$

Integrals of the form $\int (ax+b)\sqrt{px^2+qx+r} dx$

Use $ax+b = A \frac{d}{dx}(px^2+qx+r) + B$ if substitution technique is not working.

Example 59

Evaluate $\int (x+2)\sqrt{2x^2+2x+1} dx$

Solution

Let $x+2 = A \frac{d}{dx}(2x^2+2x+1) + B$

$$x+2 = A(4x+2) + B$$

$$\begin{cases} 2A+B=2 \\ 4A=1 \end{cases} \text{ then } A=\frac{1}{4} \text{ and } B=\frac{3}{2} \text{ thus } x+2 = \frac{1}{4}(4x+2) + \frac{3}{2}$$

$$\frac{1}{4} \int (4x+2)\sqrt{2x^2+2x+1} dx + \frac{3}{2} \int \sqrt{2x^2+2x+1} dx$$

Consider $\frac{1}{4} \int (4x+2)\sqrt{2x^2+2x+1} dx$, Let $u = \sqrt{2x^2+2x+1} \Rightarrow$

$$2udu = (4x+2)dx \text{ thus } \frac{2}{4} \int u^2 du = \frac{1}{2} \left(\frac{u^3}{3} \right)$$

But $u = \sqrt{2x^2 + 2x + 1}$ thus $\frac{1}{6} \left(\sqrt{2x^2 + 2x + 1} \right)^3$

Consider $\frac{3}{2} \int \sqrt{2\left(x^2 + x + 1\right)} dx$

$$\frac{3\sqrt{2}}{2} \int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} dx$$

$$\frac{3\sqrt{2}}{2} \times \left(\frac{1}{2}\right) \int \sqrt{\left(2\left(x + \frac{1}{2}\right)^2 + 1\right)} dx$$

$$\frac{3\sqrt{2}}{4} \int \sqrt{\left(2\left(x + \frac{1}{2}\right)^2 + 1\right)} dx, \text{ let } \sinh \theta = 2\left(x + \frac{1}{2}\right) = 2x + 1 \text{ thus}$$

$$dx = \frac{\cosh \theta}{2} d\theta$$

$$\frac{3\sqrt{2}}{8} \int \sqrt{\sinh^2 \theta} \cosh \theta d\theta$$

$$\frac{3\sqrt{2}}{8} \int \cosh^2 \theta d\theta \text{ thus } \frac{3\sqrt{2}}{16} \int (\cosh 2\theta + 1) d\theta$$

$$\frac{3\sqrt{2}}{32} \sinh 2\theta + \frac{3\sqrt{2}}{16} \theta + A$$

$$\text{But } \theta = \sinh^{-1}(2x + 1)$$

$$\frac{3\sqrt{2}}{32} \sinh(2 \sinh^{-1}(2x + 1)) + \frac{3\sqrt{2}}{16} \sinh^{-1}(2x + 1) + A$$

Combining the two solutions $\int (2x + 1) \sqrt{2x^2 + 2x + 1} dx$

$$= \frac{1}{6} \left(\sqrt{2x^2 + 2x + 1} \right)^3 + \frac{3\sqrt{2}}{32} \sinh(2 \sinh^{-1}(2x + 1)) + \frac{3\sqrt{2}}{16} \sinh^{-1}(2x + 1) + A$$

Example 60

Evaluate $\int \frac{x^2 + 1}{x^4 - x^2 + 1} dx$

Solution

From $\int \frac{x^2 + 1}{x^4 - x^2 + 1} dx$ Divide throughout by x^2

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$$\int \frac{x^2 \left(1 + \frac{1}{x^2}\right)}{x^2 \left(x^2 - 1 + \frac{1}{x^2}\right)} dx \text{ complete the square } \int \left(\frac{1 + \frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2} - 2 + 2} \right) dx$$

$$\int \frac{1 + \frac{1}{x^2}}{\left(x^2 + \frac{1}{x^2} - 2\right) + 1} dx \text{ thus } \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 1} dx \quad \text{and let}$$

$$u = x - \frac{1}{x} \Rightarrow du = \left(1 + \frac{1}{x^2}\right) dx$$

$$\int \frac{du}{u^2 + 1} = \tan^{-1}(u) + A \quad \text{but } u = x - \frac{1}{x}$$

$$\text{Therefore } \int \frac{x^2 + 1}{x^4 - x^2 + 1} dx = \tan^{-1}\left(x - \frac{1}{x}\right) + A$$

EXERCISE 15

Evaluate the following integrals

1. $\int (3x+2)\sqrt{x^2-5} dx$

2. $\int (x+1)\sqrt{x^2+x+1} dx$

3. $\int (2x+1)\sqrt{(2x^2+2x-10)^5} dx$

4. $\int \frac{x^2-1}{x^4+x^2-1} dx$

5. $\int (5-x)\sqrt{2x^2+4x+8} dx$

6. $\int (3x-1)\sqrt{x^2-x+2} dx$

7. $\int (7x-5)\sqrt{2x^2-5x+3} dx$

8. $\int (2x-1)(x^2+3x-1)^3 dx$

9. $\int (4x-3)(2x^2-3x)^3 dx$

10. $\int \frac{x+1}{2x^2-2} dx$

11. $\int \frac{x^2-2}{x^4+x^2+4} dx$

12. $\int (x+3)\sqrt{x+4} dx = \frac{2}{15}(x+3)^{3/2}(3x+7) + A$

SUBSTITUTION OF INVERSE OF TRIGONOMETRIC FUNCTIONS

Integrals of the form $\int \frac{1}{\sqrt{1-f(x)^2}} dx$ or $\int \frac{1}{1+(f(x))^2} dx$

Refer the derivatives of inverse of trigonometric functions

1. If $y = \sin^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
2. If $y = \cos^{-1}(x)$ then $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$
3. If $y = \tan^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{1+x^2}$
4. If $y = \sec^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$
5. If $y = \csc^{-1}(x)$ then $\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$
6. If $y = \cot^{-1}(x)$ then $\frac{dy}{dx} = -\frac{1}{1+x^2}$

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Example 61

Evaluate $\int \frac{1}{\sqrt{4-9x^2}} dx$

Solution

This integral resembles, cosine or sine inverse, any of the function can be used.

Given $\int \frac{1}{\sqrt{4-9x^2}} dx$, this can expressed as

$$\int \frac{1}{\sqrt{4\left(1-\frac{9}{4}x^2\right)}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-\left(\frac{3}{2}x\right)^2}} dx \quad \text{let } \sin \theta = \frac{3}{2}x, \text{ then } \frac{2}{3} \cos \theta d\theta = dx$$

$$\frac{1}{2} \times \frac{2}{3} \int \frac{\cos \theta}{\sqrt{1-\sin^2 \theta}} d\theta = \frac{1}{3} \int d\theta = \frac{1}{3} \theta + A, \text{ but } \theta = \sin^{-1}\left(\frac{3}{2}x\right) \text{ therefore}$$

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3} \sin^{-1}\left(\frac{3}{2}x\right) + A$$

EXERCISE 16

Evaluate the following

1. $\int \frac{dx}{\sqrt{5-3x^2}}$
2. $\int \frac{1}{\sqrt{1+2x-9x^2}} dx$
3. $\int \frac{1}{4+49x^2} dx$
4. $\int \frac{1}{\sqrt{6+x-x^2}} dx$
5. $\int \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}} dx$
6. $\int \frac{dx}{\sqrt{3+3x-x^2}}$
7. $\int \frac{1}{1+x+x^2} dx$
8. $\int \frac{dx}{(2-4x-x^2)^{1/2}}$
9. $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$
10. $\int \frac{2x+3}{\sqrt{4-x^2}} dx$

SUBSTITUTION OF INVERSE OF HYPERBOLIC FUNCTIONS

Refer the derivatives of inverse of hyperbolic functions, for further studies of these integrals of these form, check the hyperbolic functions topic in book 2.

1. If $y = \cosh^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$
2. If $y = \sinh^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$
3. If $y = \tanh^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{1-x^2}$
4. If $y = \operatorname{sech}^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{x\sqrt{1-x^2}}$
5. If $y = \operatorname{cosech}^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2+1}}$
6. If $y = \coth^{-1}(x)$ then $\frac{dy}{dx} = \frac{1}{x^2-1}$

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Example 62

Integrate $\int \frac{6x+1}{\sqrt{1-3x^2}} dx$

Solution

From $\int \frac{6x+1}{\sqrt{3x^2-1}} dx$, thus $\int \frac{6x}{\sqrt{3x^2-1}} dx + \int \frac{dx}{\sqrt{3x^2-1}}$

INTEGRAL OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Suppose we have $\int a^x dx$, for all $a > 0$

Let $y = a^x$ then $\ln y = \ln a^x \Rightarrow \ln y = x \ln a$

$\frac{1}{y} \frac{dy}{dx} = \ln a$ thus $dy = y \ln a dx \Rightarrow \int dy = \int y \ln a dx$

$y = \ln a \int y dx$ but $y = a^x$ thus $a^x = \ln a \int a^x dx$

Therefore $\int a^x dx = \frac{a^x}{\ln a} + A$, where $a > 0$

Example 63

Integrate $\int (x)(3^{x^2}) dx$

Solution

From $\int (x)(3^{x^2}) dx$, let $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

$\int x(3^y) \frac{dy}{2x} = \frac{1}{2} \int 3^y dy$, let

$\frac{1}{y} \frac{dy}{dx} = 2 \ln 3$ thus $\frac{dy}{2 \ln 3} = y dx \Rightarrow \frac{1}{2 \ln 3} \int dy = \int y dx$ replacing y

Therefore, $\int 3^{x^2} dx = \frac{3^{x^2}}{\ln 9} + A$

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Example 64

Integrate $\int (3e^x + 2^x) dx$

Solution

$$\int (3e^x + 2^x) dx = 3 \int e^x dx + \int 2^x dx \text{ thus } \int (3e^x + 2^x) dx = 3e^x + \frac{2^x}{\ln 2} + A$$

Example 65

Integrate $\int 4^{2x-1} dx$

Solution

$$\text{Let } y = 4^{2x-1} \Rightarrow \ln y = (2x-1)\ln 4$$

$$\frac{1}{y} \frac{dy}{dx} = 2\ln 4 \Rightarrow dy = y \ln 16 dx$$

$$\int dy = \ln 16 \int y dx \Rightarrow y = \ln 16 \int y dx$$

$$\text{but } y = 4^{2x-1}$$

$$4^{2x-1} = \ln 16 \int 4^{2x-1} dx \text{ therefore } \int 4^{2x-1} dx = \frac{4^{2x-1}}{\ln 16} + A$$

Example 66

Integrate $\int (10^x + 3^x - e^{2x}) dx$

Solution

Using a general observation $\int (10^x + 3^x - e^{2x}) dx$

Can be expressed as $\int 10^x dx + \int 3^x dx - \int e^{2x} dx$ thus

$$\int (10^x + 3^x - e^{2x}) dx = \frac{10^x}{\ln 10} + \frac{3^x}{\ln 3} - \frac{1}{2} e^{2x} + A$$

Question

$$\text{Show that } \int 3^{2x+2} dx = \frac{3^{2x+2}}{\ln 9} + A$$

Example 67

Integrate $\int \frac{10x^4 + 5^x \ln 5}{2x^5 + 5^x} dx$

Solution

From $\int \frac{10x^4 + 5^x \ln 5}{2x^5 + 5^x} dx$, let $u = 2x^5 + 5^x$ thus $\frac{du}{dx} = 10x^4 + 5^x \ln 5$

$$\int \frac{1}{u} du = \ln u + A \text{ thus } \int \frac{10x^4 + 5^x \ln 5}{2x^5 + 5^x} dx = \ln(2x^5 + 5^x) + A$$

DEFINITE INTEGRALS

The fundamental theorem of calculus

The integrals of the form $\int_a^b f(x) dx = F(b) - F(a)$ are called **definite**

integrals. If $a = b$ then $\int_a^b f(x) dx = F(b) - F(b) = F(a) - F(a) = 0$

Example 68

Evaluate the value of $\int_1^2 (x^3 + 2x - 3) dx$

Solution

$$\text{From } \int_1^2 (x^3 + 2x - 3) dx = \left. \frac{x^4}{4} + x^2 - 3x + A \right|_1^2$$

$$\left. \frac{x^4}{4} + x^2 - 3x + A \right|_1^2 = \left(\frac{2^4}{4} + 2^2 - 3 \times 2 + A \right) - \left(\frac{1^4}{4} + 1^2 - 3 \times 1 + A \right)$$

$$\text{Therefore } \int_1^2 (x^3 + 2x - 3) dx = 5$$

Example 69

By first expressing $\frac{4x^2 + 4x - 17}{2x^2 + 5x - 3}$ into partial fractions, show that

$$\int_1^2 \frac{4x^2 + 4x - 17}{2x^2 + 5x - 3} dx = 2 - \ln\left(\frac{45}{4}\right)$$

Solution

Use long division method first to simplify the function $\frac{4x^2 + 4x - 17}{2x^2 + 5x - 3}$

$$\begin{array}{r} 2x^2 + 5x - 3 \overline{) 4x^2 + 4x - 17} \\ \underline{-4x^2 + 10x - 6} \\ -6x - 11 \end{array}$$

$$\frac{4x^2 + 4x - 17}{2x^2 + 5x - 3} = 2 - \frac{6x + 11}{2x^2 + 5x - 3}$$

Partial fractions

$$\frac{11}{2x^2 + 5x - 3} = \frac{6x + 11}{(2x - 1)(x + 3)}$$

$$\frac{6x + 11}{(2x - 1)(x + 3)} = \frac{A}{2x - 1} + \frac{B}{x + 3} \text{ thus } 6x + 11 = A(x + 3) + B(2x - 1)$$

When $x = -3$, $B = 1$ and when $x = \frac{1}{2}$, $A = 4$

$$\frac{11}{(2x - 1)(x + 3)} = \frac{4}{(2x - 1)} + \frac{1}{(x + 3)}$$

Now combine the solutions $\int_1^2 \frac{4x^2 + 4x - 17}{2x^2 + 5x - 3} dx$

$$\begin{aligned} & 2 \int_1^2 dx - 4 \int_1^2 \frac{1}{2x - 1} dx + \int_1^2 \frac{1}{x + 3} dx \\ &= 2 - \ln \left(\frac{9 \times 5}{4} \right) \end{aligned}$$

Therefore $\int_1^2 \frac{4x^2 + 4x - 17}{2x^2 + 5x - 3} dx = 2 - \ln \left(\frac{45}{4} \right)$ Hence shown

Example 70

Evaluate $\int_0^{\frac{\pi}{3}} (\sec x + \tan x) dx$

Solution

From $\int_0^{\frac{\pi}{3}} (\sec x + \tan x) dx$ thus $\int_0^{\frac{\pi}{3}} \sec x dx + \int_0^{\frac{\pi}{3}} \tan x dx$

$$\begin{aligned}
 &= \ln|\sec x + \tan x| + \ln|\sec x| \Big|_0^{\frac{\pi}{3}} \\
 &= \ln|2 + \sqrt{3}| + \ln|2| = \ln\left|1 + \frac{\sqrt{3}}{2}\right|
 \end{aligned}$$

Therefore $\int_0^{\frac{\pi}{3}} (\sec x + \tan x) dx = \ln\left|1 + \frac{\sqrt{3}}{2}\right|$

Example 71

Express $\frac{2x+1}{(x-3)^2}$ in to partial fractions and hence show that

$$\int_4^{10} \frac{2x+1}{(x-3)^2} dx = 6 + \ln|49|$$

Solution

From $\frac{2x+1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$ and $2x+1 = Ax - 3A + B$

Compare the coefficients $A=2$ and $B=7$

$$\frac{2x+1}{(x-3)^2} = \frac{2}{x-3} + \frac{7}{(x-3)^2}$$

$$\int_4^{10} \frac{2x+1}{(x-3)^2} dx = \int_4^{10} \frac{2}{x-3} dx + \int_4^{10} \frac{7}{(x-3)^2} dx$$

$$\int_4^{10} \frac{2x+1}{(x-3)^2} dx = 2\ln|x-3| - \frac{7}{x-3} \Big|_4^{10}$$

$$\int_4^{10} \frac{2x+1}{(x-3)^2} dx = (2\ln 7 - 1) - (2\ln 1 - 7)$$

$$\therefore \int_4^{10} \frac{2x+1}{(x-3)^2} dx = 6 + \ln|49| \text{ Hence shown}$$

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Example 72

Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x} = \frac{\pi}{4}$

Solution

Let $I_n = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \tan x}$

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\cos x dx}{\cos x + \sin x}$$

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right) dx}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} = \int_0^{\frac{\pi}{2}} \frac{\sin x dx}{\cos x + \sin x}$$

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin x dx}{\cos x + \sin x}$$

$$2I_n = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx$$

$$2I_n = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\cos x + \sin x} dx$$

$$2I_n = x \Big|_0^{\frac{\pi}{2}} \text{ then } 2I_n = \frac{\pi}{2} \text{ therefore } I_n = \frac{\pi}{4}, \text{ proved}$$

EXERCISE 17

Evaluate the following

1. $\int_1^8 \left(\sqrt{x} + \frac{3}{\sqrt[3]{x}} + 4 \right) dx$

2. $\int_1^2 (2x+1)(2x^2+2x-10)^4 dx$

3. $\int_1^5 \frac{x+2}{\sqrt{3x^2+12x-5}} dx$

4. $\int_{-\pi/2}^{\pi/2} (\sin 2x) dx$

5. $\int_1^4 \frac{1}{(x+3)^2} dx$

6. $\int_1^2 \frac{x+1}{(x+2)(x+3)} dx$

7. $\int_0^{\pi/3} (\cos 3x \cos x) dx$

8. $\int_{1/7}^{\sqrt{3}/7} \left(\frac{1}{1+49x^2} \right) dx$

9. $\int_{\pi/4}^{\tan^{-1}(2)} \left(\frac{3}{4 \cos 2x - 3 \sin 2x} \right) dx$

10. $\int_0^{1.1} (\sec^3 x \tan x) dx$

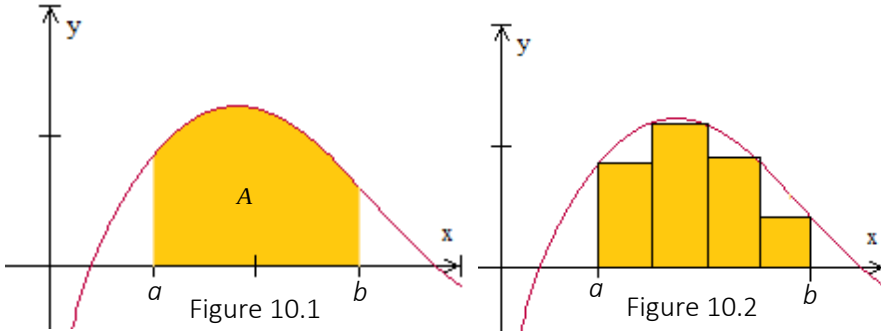
11. $\int_2^3 \left(\frac{\ln x}{x} \right) dx$

APPLICATIONS OF INTEGRATION

There are several applications of integration.

AREA UNDER THE CURVE: $\text{Area} = \int_a^b y dx$

Suppose we want to find the area under the curve $y = f(x)$ sketched below bounded from $x = a$ to $x = b$ with the curve.



One way to estimate the area, A , (Figure 10.1) is to divide the area into rectangular strips, (Figure 10.2) the sum of the areas of strips is approximated to the area under the curve. The thinner the strips the better approximation, i.e. as the number of strips increase the better approximation of the area under the curve.

Since each strip have the same width, δx the length of each strip is y , therefore, the area of one strip is given as $\delta A \approx y \delta x$, to get the area under the curve we sum the area of all strips from $x = a$ to $x = b$, this can be

written as $\text{Area} = \sum_{x=a}^b \delta A$, as the width of strips decrease, the number of

rectangles increase, therefore, as $\delta x \rightarrow 0$, $\frac{\delta A}{\delta x} \approx y \rightarrow \lim_{\delta x \rightarrow 0} \left(\frac{\delta A}{\delta x} \right) = \frac{dA}{dx}$

Then $\frac{dA}{dx} = y$ becomes $dA = y dx \Rightarrow A = \int y dx$, the boundary limit of x is

$x = a$ to $x = b$. This defines the area under the curve as, $A = \int_a^b y dx$.

Procedures to find the area

- Draw a sketch of the given region to visualize the nature of the curve and divide intervals appropriately to avoid getting negative or zero answer.
- Write the defined integral(s) to represent the region

(c) Evaluate the defined integral.

Example 70

Determine the area under the curve $y = -x + 3$ from $x = 0$ to $x = 2$

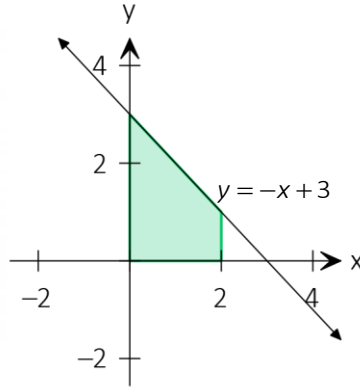
Solution

Sketch

$$A = \int_0^2 (-x + 3) dx$$

$$A = \left(-\frac{x^2}{2} + 3x \right) \Big|_0^2 \text{ thus } A = 4$$

The area is four sq. units



Example 71

Find the area under the curve $y = -x^2 + 4$ from $x = 1$ to $x = 2$

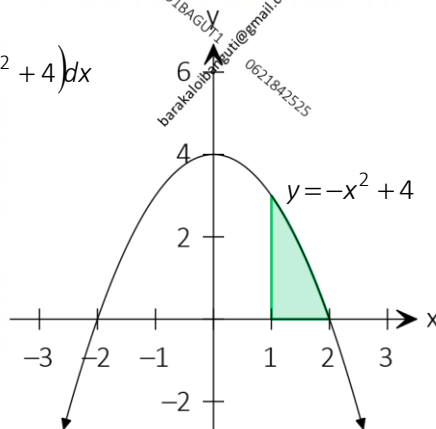
Solution

$$\text{Area} = \int_a^b y dx \text{ therefore } A = \int_1^2 (-x^2 + 4) dx$$

$$A = -\frac{x^3}{3} + 4x \Big|_1^2$$

$$A = \left(-\frac{8}{3} + 8 \right) - \left(-\frac{1}{3} + 4 \right)$$

The area is $\frac{5}{3}$ square units.



Example 72

Evaluate the area of the curve $y = x^3 + 2x^2$ from $x = -1$ to $x = 1$

Solution

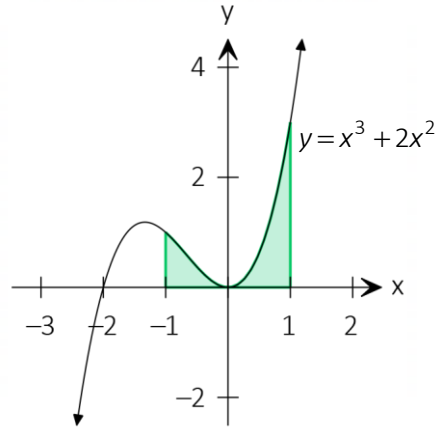
$$A = \int_{-1}^1 (x^3 + 2x^2) dx$$

$$A = \int_{-1}^0 (x^3 + 2x^2) dx + \int_0^1 (x^3 + 2x^2) dx$$

$$A = \left(\frac{x^4}{4} + \frac{2x^3}{3} \right) \bigg|_{-1}^0 + \left(\frac{x^4}{4} + \frac{2x^3}{3} \right) \bigg|_0^1$$

$$A = -\left(\frac{1}{4} - \frac{2}{3} \right) + \left(\frac{1}{4} + \frac{2}{3} \right) = 1.3$$

The area is 1.3 square units.

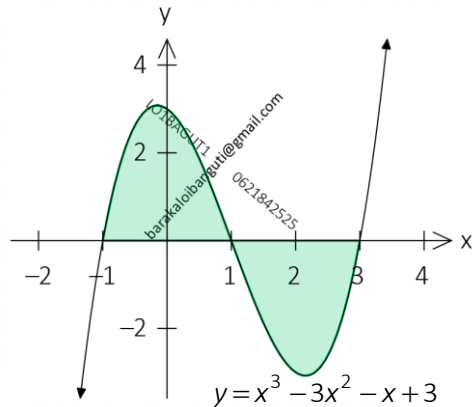


Example 73

Determine the area of the curve $y = x^3 - 3x^2 - x + 3$ from $x = -1$ to $x = 3$

Solution

From the sketch, a part of the integral lies below the x - axis and the other part lie above, if we evaluate the integral direct from $x = -1$ to $x = 3$, the result will be zero, we divide the integral as follows



$$A = \int_{-1}^1 (x^3 - 3x^2 - x + 3) dx + \int_1^3 (x^3 - 3x^2 - x + 3) dx$$

While dividing it, we should make sure the area obtained is always positive, this can be done by taking the absolute value of each integral above.

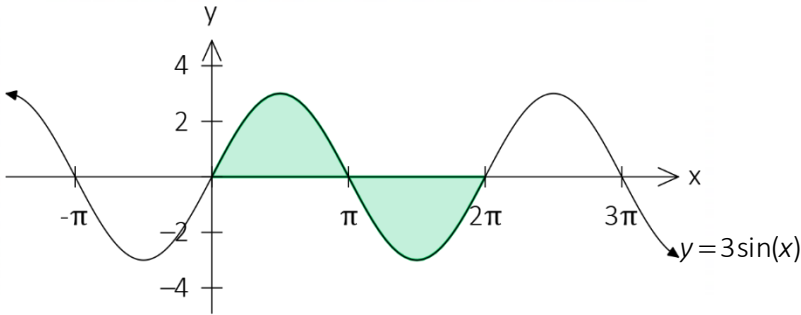
$$A = \left| \left(\frac{x^4}{4} - x^3 - \frac{x^2}{2} + 3x \right) \bigg|_{-1}^1 \right| + \left| \left(\frac{x^4}{4} - x^3 - \frac{x^2}{2} + 3x \right) \bigg|_1^3 \right|$$

The area is 4.5 square units.

Example 74

Find the area of the curve $y = 3\sin x$ from $x = 0$ to $x = 2\pi$

Solution



$$A = \int_a^b y dx \text{ then } A = \int_0^{2\pi} (3 \sin x) dx$$

$$A = \int_0^{\pi} (3 \sin x) dx + \int_{\pi}^{2\pi} (3 \sin x) dx$$

$$A = (3 \cos x) \Big|_0^{\pi} + (3 \cos x) \Big|_{\pi}^{2\pi}$$

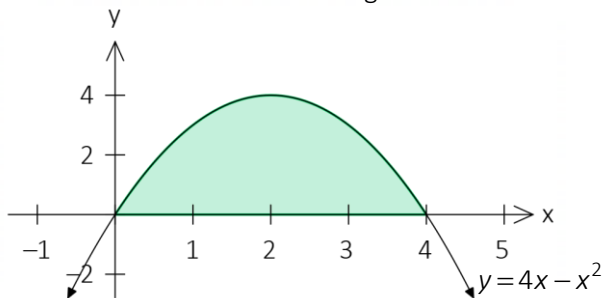
$$A = |3 \cos \pi - 3 \cos 0| + |3 \cos 2\pi - 3 \cos \pi|$$

$$A = 12 \text{ Square units.}$$

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Example 75

Find an approximate area of the shaded region below



Solution

$$A = \int_0^4 (4x - x^2) dx \quad \text{thus} \quad A = 2x^2 - \frac{x^3}{3} \Big|_0^4$$

The area is $A = \frac{32}{3}$ square units.

AREA BETWEEN CURVES: $A = \int_a^b (y_1 - y_2) dx$ (where a and b are abscissa at a

point of intersection)

If the area under the curve $y = f(x)$ is

given by $A_1 = \int_a^b f(x) dx$ and the area

under the curve $y = g(x)$ is given as

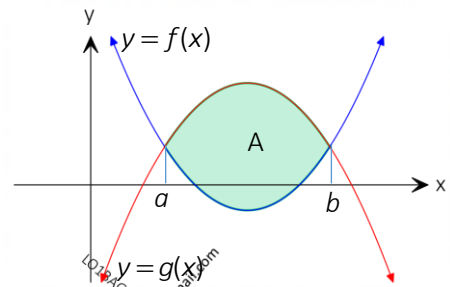
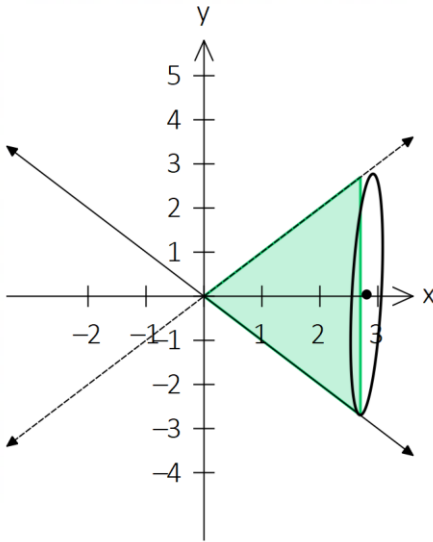


Figure 10.3



$$A_2 = \int_a^b g(x) dx, \text{ then the area}$$

between the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ (Where a and b are the values of x at the point of intersection of the curves) is given by

$$A = \int_a^b [f(x) - g(x)] dx, \text{ (Figure 10.3)}$$

Example 76

Find the area between curves $y = 4 - x^2$ and $y = x^2 - 2x$

Solution

First, find the points of intersection of the curves, $x^2 - 2x = 4 - x^2$

$$2x^2 - 2x - 4 = 0$$

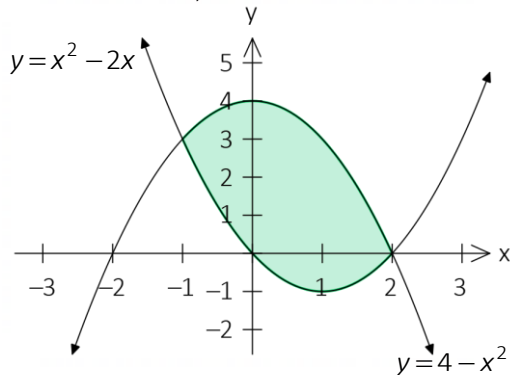
$$(x + 1)(x - 2) = 0$$

$$x = -1 \text{ and } x = 2$$

$$A = \left| \int_a^b (y_2 - y_1) dx \right|$$

$$A = \int_{-1}^2 ((4 - x^2) - (x^2 - 2x)) dx$$

$$A = \int_{-1}^2 (4 - 2x^2 + 2x) dx$$



$$A = 4x - \frac{2}{3}x^3 + x^2 \Big|_{-1}^2 \quad \text{therefore the area is 9 square units}$$

Example 77

Find the area enclosed by the curves $y = 2x^2 + 7x + 3$ and $y = 9 + 4x - x^2$

Solution

The points of intersection of the curve

$$2x^2 + 7x + 3 = 9 + 4x - x^2 \Rightarrow (x - 1)(x + 2) = 0 \quad \text{thus } x = 1 \text{ or } x = -2$$

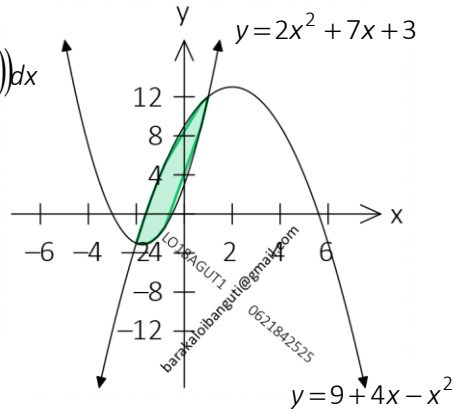
$$A = \int_a^b (y_2 - y_1) dx$$

$$A = \int_{-2}^1 ((2x^2 + 7x + 3) - (9 + 4x - x^2)) dx$$

$$A = \int_{-2}^1 (3x^2 + 3x - 6) dx$$

$$A = x^3 + \frac{3}{2}x^2 - 6x \Big|_{-2}^1$$

$$A = \frac{27}{2} \text{ square units}$$

**Example 78**

Find the area between curves $y = x^2 - 1$ and $y = 1 - x$

Solution

Point of intersections

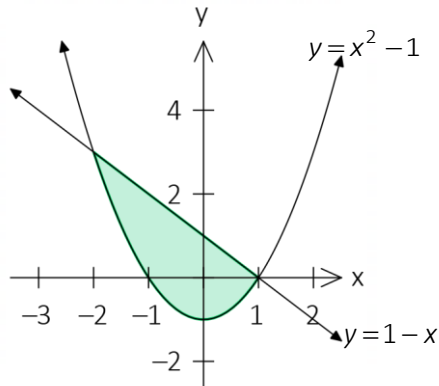
$$-x + 1 = x^2 - 1$$

$$x = -2 \text{ and } x = 1$$

$$A = \int_a^b (y_2 - y_1) dx$$

$$A = \int_{-1}^2 ((x^2 - 1) - (x + 1)) dx$$

$$A = \int_{-1}^2 (x^2 - x - 2) dx$$



$$A = \frac{x^3}{3} - \frac{x^2}{2} - 2x \Big|_{-1}^2$$

Area is 4.5 square units

Example 79

Find the area of the curve $y = 3x^4 - 6x^3 - 3x^2 + 6x$ with x – axis

Solution

The points of intersection

$$y = 3x^4 - 6x^3 - 3x^2 + 6x$$

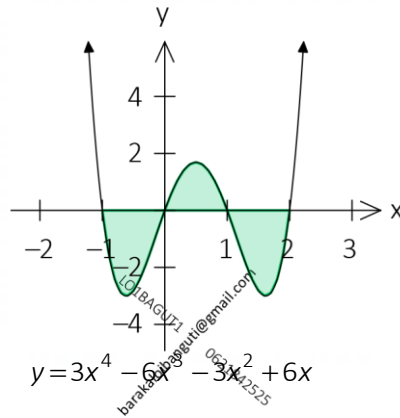
$$3x(x-3)(x-1)(x+1) = 0$$

$$x = 0, x = 3, x = 1 \text{ and } x = -1$$

$$A = \int_{-1}^3 (3x^4 - 6x^3 - 3x^2 + 6x) dx$$

$$A = \frac{3x^5}{5} - \frac{3}{2}x^4 - x^3 + 3x^2 \Big|_{-1}^3$$

$$A = 22\frac{2}{5} \text{ sq. units}$$

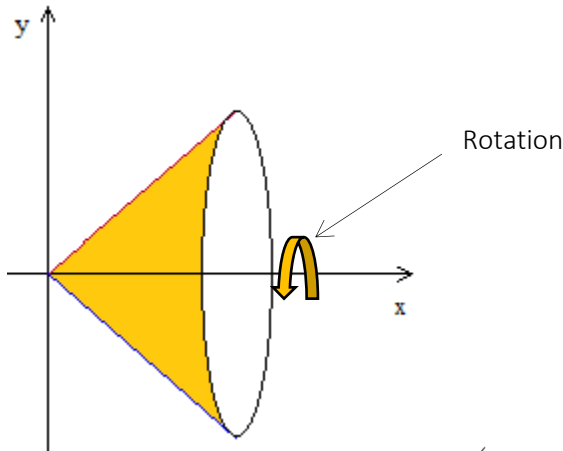


EXERCISE 18

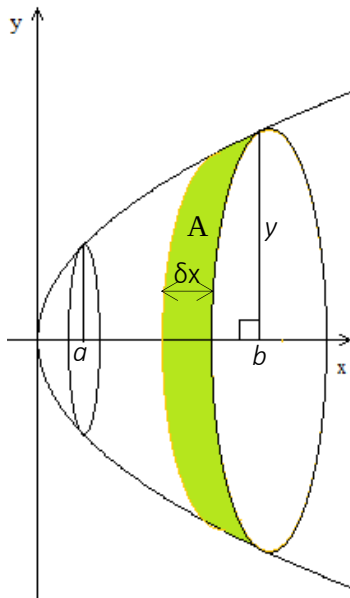
1. Find the area under the curve $f(x) = x^3 - 4x$ from $x = -2$ to $x = 2$.
2. Find the area enclosed by the x - axis and that part of the curve $f(x) = 2 - 3x + x^2$.
3. Sketch the graph of $y = \ln(x) + x^2$ and hence calculate the area bounded with the x axis from $x = 1$ to $x = 2$.
4. Find the area of the region bounded by the curve $y = 3x^2 - 2x + 1$, the line $x + 1 = 0$, $x - 2 = 0$ and $y = 0$
5. The area between the curve $3x^2 + y^2 = 9$ and the y -axis from $y = -3$ and $y = 3$ is rotated about the y -axis. Find the volume of the solid generated.
6. Find the area between the curve $y = x + 1$ and $y = 3 - x^2$
7. Find the area between the curves $x = y^2$ and $x = 3y + 4$
8. Find the area bounded by the curve $y = 1 + \sqrt{x}$ and $y = x - 1$
9. Find the area between the coordinates axes and the curve $y = \sqrt{x} - 2$
10. Find the area under the curve $y = 4$ between the lines $x = 0$ and $x = 5$
11. Find the area under the curve $\frac{y}{x+1} = 3$ from $x = 0$ to $x = 40$
12. Find the area between curves $y = x^2$ and $y = -x^2 + 2$
13. Find the value of a if the area between the curve $y = ax^2 + 2$ and $y = x^2$ from $x = 1$ to $x = 2$ is 10 square units.
14. Find the area between the curves $y^2 = 4x$ and $x^2 + y^2 = 8x$ bounded with the x - axis.
15. Find the area under the curve $x^2y + 3y = 5$ from $x = 0$ to $x = \sqrt{3}$

VOLUME OF THE SOLID OF REVOLUTION

Suppose the area enclosed by the line $y = ax$ with x -axis and the line $x = c$ is rotated through one revolution, the solid formed is called **solid of revolution**, which is cone in this case.



Consider the figure below



Volume of disc A , $V = \pi r^2 h$

To get the volume of the figure from $x = a$ to $x = b$ is the summation of the approximated volume of the discs, given $r = y$ and $h = \delta x$ therefore volume,

$$V = \sum_{x=a}^{x=b} \pi y^2 \delta x.$$

$$V = \lim_{\delta x \rightarrow 0} \left(\sum_{x=a}^{x=b} \pi y^2 \delta x \right) = \int_a^b \pi y^2 dx$$

Therefore the volume is given by

$$V = \pi \int_a^b y^2 dx$$

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The volume is given by $V = \pi \int_a^b y^2 dx$, if the curve is revolved about x – axis

or $V = \pi \int_a^b x^2 dy$ if the curve is revolved about y – axis.

Note that, if the curve is symmetrical about the y – axis or x – axis and the same curve is made to revolve about the x – axis or y – axis, we revolve only the part of the curve lying one side of either the x – axis or y – axis. To get the volume generated by considering the revolution of both sides is given by

$$V = 2\pi \int_a^b y^2 dx.$$

Example 80

Find the volume of the solid of revolution formed by rotating the area enclosed by the curve $y = x^2 + x$, the x – axis and the ordinates $x = 1$ and $x = 2$ through one revolution about the x – axis.

Solution

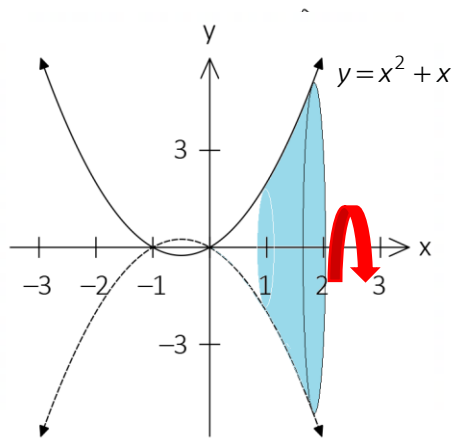
$$V = \pi \int_a^b y^2 dx$$

$$V = \pi \int_1^2 (x^2 + x)^2 dx$$

$$V = \pi \int_1^2 (x^4 + 2x^3 + x^2) dx$$

$$V = \pi \left(\frac{1}{5} x^5 + \frac{1}{2} x^4 + \frac{1}{3} x^3 \right) \Big|_1^2$$

$$V = \frac{481}{30} \pi$$



Therefore, the volume is 50.37 cubic units

Example 81

Find the volume of the solid formed when rectangular hyperbola $xy = 8$ rotated once through one revolution with positive x – axis from $x = 2$ and $x = 5$

Solution

$$xy = 8 \text{ then } y = \frac{8}{x}$$

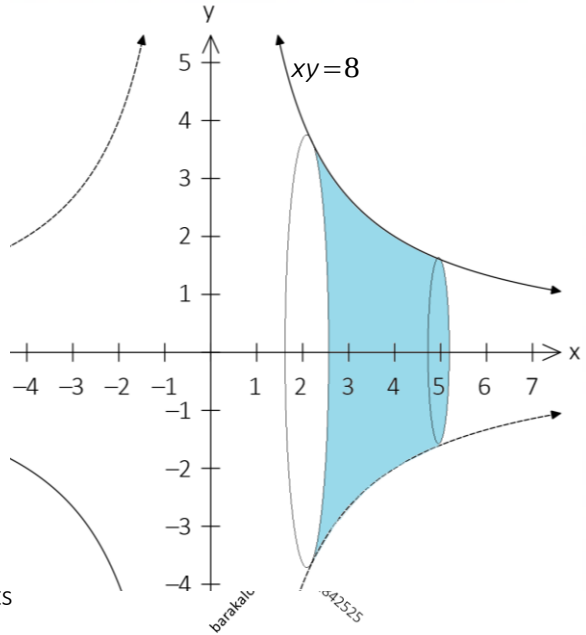
$$V = \pi \int_2^5 \left(\frac{8}{x} \right)^2 dx$$

$$V = \pi \int_2^5 \left(\frac{64}{x^2} \right) dx$$

$$V = -\pi \left(\frac{64}{x} \right) \Big|_2^5$$

$$V = \frac{96}{5} \pi$$

The volume is 60.32 cubic units



Example 82

Find the volume in the first quadrant bounded by curve $y = 2x^2 + 1$, the y – axis and the lines $y = 2$ and $y = 5$.

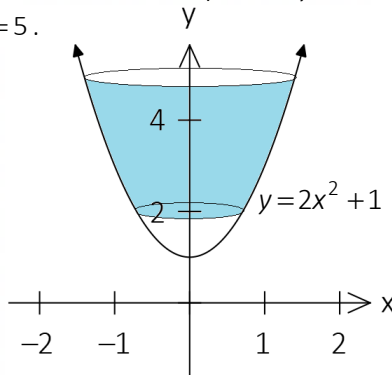
Solution

$$y = 2x^2 + 1$$

$$x = \sqrt{\frac{y-1}{2}}$$

$$V = \pi \int_2^5 \left(\sqrt{\frac{y-1}{2}} \right)^2 dy$$

$$V = \frac{\pi}{2} \int_2^5 (y-1) dy$$



$$V = \frac{\pi}{2} \left(\frac{y^2}{2} - y \right) \Bigg|_2^5$$

$$V = \frac{15}{4} \pi$$

The volume is 11.78 cubic units

EXERCISE 19

Evaluate the following

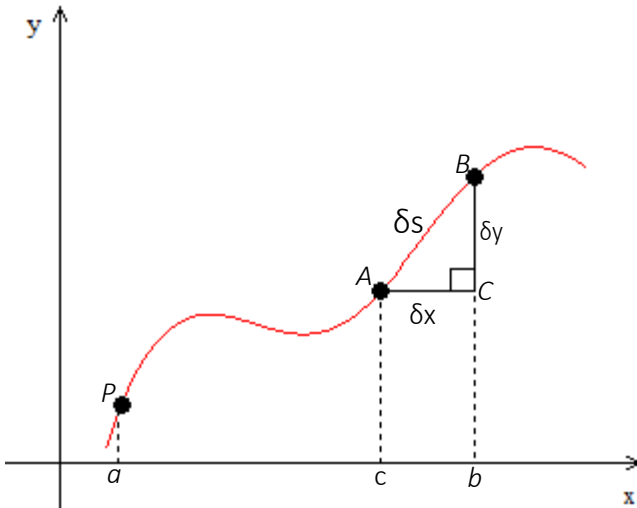
1. A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y -axis. Find the volume of the bowl.
2. Find the volume of solid revolution when a curve $xy = 7$ is rotated about the x axis from $x = 3$ to $x = 5$, leave your answer with π .
3. Calculate the volume generated when the curve $x^2y = x^4 + 3$ is rotated once about x -axis from $x = 1$ to $x = 2$.
4. Find the volume generated when the loop of the curve $4x^2 = y(5 - y)^2$ revolve about the y -axis.
5. Find the volume generated by the rotation of the curve $x^2y + 3y = 5$ about y - axis, from $y = 0.5$ to $y = 1.6$.
6. Find the volume generated by revolving the curve $y = \cos(x)$ from $x = -\frac{3}{2}\pi$ to $x = \frac{3}{2}\pi$ along the x - axis once.
7. Find the volume generated by revolving the curve $y = \sqrt{x + 4}$ from $x = 1$ to $x = 2$ about the x - axis once.
8. The curve $y^2 + 3 - 2x = 0$ is rotated about x - axis once, from $x = 2$ to $x = 4$.
9. The bowl in a shape of parabola open upward has the equation $x^2 = 2(3 + y)$ is filled with water, find the volume of water contained in a bowl in cubic centimetres if the vertical height of the bowl is 8 units from the bottom and 4 cubic units = 15 cm³.
10. Find the volume generated by revolving the curve $\frac{5y}{4x + 4} = \pi$ along line $y = -1$ from $y = 0$ to $y = 4$.

11. Find the volume formed when the loop of the curve $\frac{3}{y} = \left(\frac{3-y}{x}\right)^2$ revolve about the y – axis.
12. Show that the volume formed by the curve $x^2 + y^2 = 9$ is a same as the volume of the sphere with radius 3 units.

ARC LENGTH

There is no simple formula to calculate length of the curve other than a circle. If the length of the portion of a curve is needed we use the method of summing small portions of the required curve to the end of the required portion. Suppose that the arc AB of the length δs is such an element, then

the length s of the curve PB is given by $s = \sum_{x=a}^{x=b} \delta s$



The length of the curve AB , where A and B are close to each other, then

By Pythagoras theorem

$$(\delta s)^2 \approx (\delta x)^2 + (\delta y)^2$$

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

$$\delta s = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x$$

$$s = \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x$$

As $\delta x \rightarrow 0$ then $\frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}$

$$s = \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \delta x$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

The length of an arc is given by the formula, $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

Example 83

Find the length of the portion of the curve $y = x^2$ between $x = 0$ and $x = 1$

Solution

$$\text{From } y = x^2 \Rightarrow \frac{dy}{dx} = 2x$$

$$s = \int_0^1 \sqrt{1 + (2x)^2} dx$$

Use hyperbolic substitution

$$\text{Let } 2x = \sinh \theta \Rightarrow 2dx = \cosh \theta d\theta$$

$$s = \frac{1}{2} \int_1^2 \sqrt{1 + \sinh^2 \theta} \cosh \theta d\theta$$

$$s = \frac{1}{2} \int_1^2 \cosh^2 \theta d\theta$$

Remember $\cosh 2\theta = \cosh^2 \theta + \sinh^2 \theta$ and $\cosh^2 \theta - \sinh^2 \theta = 1$

$$\cosh 2\theta = 2 \cosh^2 \theta - 1$$

$$\cosh^2 \theta = \frac{1}{2} (\cosh 2\theta + 1)$$

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$$s = \frac{1}{2} \int_0^1 \cosh^2 \theta d\theta$$

$$s = \frac{1}{4} \int_0^1 (\cosh 2\theta + 1) d\theta$$

$$s = \frac{1}{4} \left(\frac{1}{2} \sinh 2\theta + \theta \right)$$

$$s = \frac{1}{4} (\sinh \theta \cosh \theta + \theta)$$

$$s = \frac{1}{4} \left(2x\sqrt{1+4x^2} + \sinh^{-1}(2x) \right)$$

$$s = \frac{1}{4} \left(2x\sqrt{1+4x^2} + \ln \left| 2x + \sqrt{(2x)^2 + 1} \right| \right) \Bigg|_0^1$$

$$s = \frac{1}{4} (2\sqrt{5} + \ln |2 + \sqrt{5}|)$$

$$s = \frac{1}{2} \sqrt{5} + \frac{1}{4} \ln(2 + \sqrt{5})$$

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The length is 1.48 unit length

Example 84

Find the length of the arc $y = x^{\frac{3}{2}}$ from $x = 1$ to $x = 8$ correct to 1 decimal place.

Solution

$$\text{From } y = x^{\frac{3}{2}} \Rightarrow \frac{dy}{dx} = \frac{3}{2} x^{\frac{1}{2}}$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

$$s = \int_1^8 \sqrt{1 + \left(\frac{3}{2} x^{1/2} \right)^2} dx$$

The length is 22.8 units

Find the length of the arc of $y = \ln|\sec x|$ from $x = 0$ to $x = \frac{\pi}{3}$

$$\text{From } y = \ln|\sec x| \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x$$

$$s = \ln|2 + \sqrt{3}| \approx 1.32 \text{ units}$$

Find the length of an arc of the curve $y = e^{2x} - x$ from $x = 3$ to $x = 10$

$$y = e^{2x} - x \text{ then } \frac{dy}{dx} = 2e^{2x} - 1$$

$$s = \sqrt{2}e^3(e^7 - 1)$$

Example 87

$$y = \frac{1}{6}x^3 + \frac{1}{2x} \text{ from } x=1 \text{ to } x=2$$

x from $x = 3$

From $y = \frac{1}{6}x^3 + \frac{1}{2x}$ then $\frac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2x^2}$

$$s = \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2x^2}\right)^2} dx$$

$$s = \int_1^2 \sqrt{1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4x^4}} dx$$

$$s = \int_1^2 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)^2} dx$$

$$s = \int_1^2 \left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right) dx$$

$$s = \left. \frac{x^3}{6} - \frac{1}{2x} \right|_1^2$$

The length is 1.417 unit correct to 4 significant figures.

Example 88

Find the length of the arc of the curve $x^2 + y^2 = a^2$ between the points where $x = a \cos \alpha$ and $x = a \cos \beta$

Solution

From $x^2 + y^2 = a^2 \Rightarrow 2x + 2y \frac{dy}{dx} = 0$ therefore $\frac{dy}{dx} = -\frac{x}{y}$

$$s = \int_a^b \sqrt{1 + \left(-\frac{x}{y}\right)^2} dx$$

$$s = \int_{a \cos \alpha}^{a \cos \beta} \left(\sqrt{\frac{y^2 + x^2}{y^2}} \right) dx \text{ thus } s = \int_{a \cos \alpha}^{a \cos \beta} \left(\sqrt{\frac{a^2}{y^2}} \right) dx$$

$$s = \int_{a \cos \alpha}^{a \cos \beta} \left(\frac{a}{y} \right) dx \text{ But } y = \sqrt{a^2 - x^2}$$

$$s = \int_{a \cos \alpha}^{a \cos \beta} \left(\frac{a}{\sqrt{a^2 - x^2}} \right) dx \text{ thus } s = \int_{a \cos \alpha}^{a \cos \beta} \left(\frac{a}{\sqrt{a^2 - x^2}} \right) dx$$

Now let $x = a \cos \theta \Rightarrow dx = -a \sin \theta d\theta$

$$s = \int_{a \cos \alpha}^{a \cos \beta} \frac{-a^2 \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} d\theta$$

$$s = -a \int_{a \cos \alpha}^{a \cos \beta} d\theta \text{ thus then } s = -a\theta \text{ and } \theta = \cos^{-1}\left(\frac{x}{a}\right)$$

$$s = -a \cos^{-1}\left(\frac{x}{a}\right) \Big|_{a \cos \alpha}^{a \cos \beta}$$

$$s = \left(-a \cos^{-1}\left(\frac{a \cos \alpha}{a}\right)\right) - \left(-a \cos^{-1}\left(\frac{a \cos \beta}{a}\right)\right)$$

$$s = -a\alpha + a\beta$$

The length is $a(\beta - \alpha)$ units

PARAMETRIC LENGTH OF THE CURVE

For the parametric equations of the curve

$$x = f(t) \Rightarrow \frac{dx}{dt} = f'(t) \text{ and } y = g(t) \Rightarrow \frac{dy}{dt} = g'(t)$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The length of the curve given in parametric is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 89

Determine the length of the parametric curve given by the following parametric equations, $y = 3 \sin t$ and $x = 3 \cos t$ from $t = 0$ to $t = 2\pi$

Solution

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From $y = 3 \sin t \Rightarrow \frac{dy}{dt} = 3 \cos t$ and $x = 3 \cos t \Rightarrow \frac{dx}{dt} = -3 \sin t$

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$s = \int_0^{2\pi} \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} dt$$

$$s = 3 \int_0^{2\pi} dt \quad \text{therefore } s = 3t \Big|_0^{2\pi}$$

$s = 6\pi$, the length of the arc is 18.85 unit length

EXERCISE 20

- Calculate the arc length of the curve $y^2 = 4 - x^2$ from $x = -2$ to $x = 2$
- Calculate the arc length of the curve $f(x) = x^2 - x$ from $x = 1$ to $x = 2$
- Show that the length of the curve $e^y = \cos x$ from $x = 0$ to $x = \frac{\pi}{4}$ is $s = \ln(1 + \sqrt{2})$
- Show that the length of the curve $y = \cosh x$ from $x = 0$ to $x = 1$ is $\ln\left(\frac{e^2 - 1}{2}\right) - 1$
- Find the length of the curve $y^2 = x^3$ from $x = 0$ to $x = 1$ correct to 4 significant figures.
- Compute the length of the curve $y^2 = \sqrt{2}(x - 1)$ from $x = 1$ to $x = 5$
- Find the length of the curve $y = \ln(\tan(x))$ from $x = 1$ to $x = 3$
- Find the length of the curve $x = 2 \cos 3u$, $y = 2 \sin 3u$ from $u = \pi$ to $u = 2\pi$
- Find the length of the curve given parametrically as $x = 3t^2 + 2$, $y = 5t^2 - 3$ from the point where $t = 3$ to $t = 8$
- Determine the length of the curve given as $p = r^2 - 4r$, $q = 3r - 2r^2$ from $r = 4$ to $r = 10$

MISCELLANEOUS EXERCISE

Evaluate the following

1. $\int \frac{e^x - 1}{e^x - x} dx$
2. $\int \left(\frac{6+x}{x^2+9} \right) dx$
3. $\int \frac{1}{3x+9} dx$
4. $\int \frac{1}{y(y-3)} dy$
5. $\int x(2^x) dx$
6. $\int x \cos\left(\frac{x}{2}\right) dx$
7. $\int \left(\frac{2x-1}{4x^2+5} \right) dx$
8. $\int \frac{\cos x - 2 \sin x}{3 \cos x + 4 \sin x} dx$
9. $\int \frac{1-4x}{\sqrt{1+2x-x^2}} dx$
10. $\int x \ln(2x) dx$
11. $\int x^5(1-x^6) dx$
12. $\int \frac{1}{1-5 \sin^2 x} dx$
13. $\int \frac{4}{\cos^2 x + 9 \sin^2 x} dx$
14. $\int \frac{1+\sin x}{\cos^2 x} dx$
15. $\int x^3 \sin x dx$
16. $\int x^5 e^x dx$
17. $\int x^5 \cos x dx$
18. $\int \frac{1}{y \sqrt{\ln y}} dy$
19. $\int \frac{x}{\sqrt{1-3x}} dx$
20. $\int x \cos 2x \sin 2x dx$
21. $\int \cos^6 x \sin^5 x dx$
22. $\int \sin^{-1}(x) dx$
23. $\int \frac{1}{\sin \theta + 2 \cos \theta + 1} d\theta$
24. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$
25. $\int \frac{dx}{(x-1)^2(x^2+1)}$
26. $\int \frac{2x^3}{x^2+x-20} dx$
27. $\int x^2 \tan^{-1} x dx$
27. $\int \frac{\ln x}{(1+x)^2} dx$
28. $\int x \ln(x+4) dx$
29. $\int \frac{\sec^2 x}{1+2 \tan x} dx$
30. $\int e^{x-|x|} dx$
31. $\int_0^4 \sin^{-1}\left(\frac{x}{4}\right) dx$

32. $\int_1^3 \frac{x^2}{x^3+1} dx$
33. $\int \frac{dx}{\sqrt{x-2} + \sqrt{x+3}}$
34. Prove that $\int_0^1 \frac{x \ln(1+x)}{1+x} dx = \frac{1}{2} \ln 2 (4 - \ln 2) - 1$
35. Evaluate (a) $\int \frac{2x-1}{\sqrt{4x-4x^2}} dx$ (b) $\int (\sec x + \tan x) dx$
36. Express $\frac{1}{1-x^4}$ in partial fractions and hence show that e
- $$\int_0^{1/2} \frac{1}{1-x^4} dx = \frac{1}{4} \ln 3 + \frac{1}{3} \tan^{-1} \left(\frac{1}{2} \right)$$
37. Evaluate $\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)} dx$ express your answer in the form of $\ln \left| \frac{a}{b} \right|$ where a and b are non-zero positive integers.
38. Prove that $\int_0^{\pi/2} \frac{\sin x}{3 \sin x + 4 \cos x} dx = \frac{3}{50} \pi + \frac{4}{25} \ln \left(\frac{4}{3} \right)$
39. By making the substitution $x = \pi - y$, or otherwise, prove that $\int_0^{\pi} x \sin^3 x dx = \frac{2}{3} \pi$
40. Use the substitution $x = \pi - x$ show that $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$
41. Find the area enclosed by the curve $f(x) = x^3 - 3x^2 + 2x$ and the x -axis between $x=0$ and $x=2$
42. Find the area enclosed between the curve $f(x) = \sin 3x$ and the x -axis between $x = -\pi$ and $x = \pi$
43. Find the area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$
44. Find the area between curves $x^2 + y^2 = 1$ and $3x^2 + y^2 = 3$
45. Find the area enclosed between curve $y = -3x + x^2$ and the line $y = 4$
46. Find the volume of the solid revolution formed when curve $y = x^2 - 3$ from $x=1$ to $x=3$
47. Find the length of the arc of the curve $3y = \sin 3x$ between the points $x = \pi$ and $x = 2\pi$
48. Find the area of the surface formed by the rotation of the curve $y^2 = 8x$ about the x -axis, from the origin to $x = 2$

49. Find the volume of the solid revolution obtained by rotating the region bounded by the curve $y^2 = x$, $x \geq 0$, the x -axis and the line $x=0$ and $x=2$ respectively about the x -axis.
50. Find the volume of solid of revolution obtained by rotating the region bounded by the line $y = \frac{1}{2}x$, the x -axis and the line $x=0$ and $x=4$ respectively about the x -axis.
51. Find the volume of the sphere obtained by rotating the upper-half of the circle $x^2 + y^2 = a^2$ about the x -axis.
52. Show that the volume of the solid of revolution obtained by rotating the region bounded by the curve $y = x^3$, the y -axis and the lines $y=0$ and $y=2$ about the y -axis is $V = \frac{6\sqrt[3]{4}\pi}{5}$ cubic units.
53. Show that $\int_0^1 e^{(ax+b)} dx = \frac{e^b}{a} (e^a - 1)$
54. Find the value of n so that $\int_1^2 nx(2-x^2)^2 dx = 1$
55. Find the area of the region between the curve $y = 4 - x^2$ from $x=0$ to $x=3$ and the x -axis.
56. Evaluate
- $\int \ln(3x+1) dx$
 - $\int \frac{dx}{(1+e^{2x})(1-e^{-2x})}$
57. (a) Find the following (i) $\int \frac{2-5\cos^3 x}{3\cos^2 x} dx$ (ii) $\int x\sqrt{9+x^2} dx$
- (b) Show that $\int_3^4 \frac{x^2+4}{x^2-4} dx = 1 + 2\ln\left(\frac{5}{3}\right)$
- (c) Use partial fractions to find $\int \frac{1}{x^2-2x} dx$
58. (a) Find $\int x \ln x dx$ and hence evaluate $\int_1^2 x \ln x dx$
- (b) Find $\int \frac{6+7z}{2-z} dz$ when $y(3)=1$

59. (a) Show that $\int_0^{\frac{1}{4}\pi} \cos 2x dx = \frac{1}{2}$
- (b) Express $\frac{4x}{(3x+1)(x+1)^2}$ in partial fractions
- (c) Hence show that $\int_0^1 \frac{4x}{(3x+1)(x+1)^2} dx = 1 - \ln 2$
60. Find the value of $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} (2 \sin 3x \cos 2x) dx$
61. If n is a non-zero integer and m any constant not equal to n , show that $\int_0^{\frac{1}{2}\pi} \cos(2mx) \cos(2nx) dx = \frac{m(-1)^n}{2(m^2 - n^2)} \sin \pi n$.
62. Evaluate $\int_0^{0.5\pi} \cos^4(x) \cos(4x) dx$
63. Show that
- (a) $\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + A$
- (b) $\int_0^{\frac{1}{2}\pi} x \sin^2 x dx = \frac{1}{16}(\pi^2 + 4)$
64. Show that $\int_0^{\frac{1}{3}\pi} \sin^3 \theta \cos^2 \theta d\theta = \frac{47}{480}$
65. Integrate $\int \frac{dx}{4 \cos^2 x - 9 \sin^2 x}$
66. Given that $\frac{dy}{dx} = (y+2)\sec^2 x$ and that $y(0) = \pi/4$, find an expression for y in terms of x .
67. (a) The curves $y = 3 \sin x$ and $y = 4 \cos x$ ($0 \leq x \leq \frac{1}{2}\pi$) intersect at the point A , and meet the x -axis at the origin O and the point $B(\frac{1}{2}\pi, 0)$ respectively. With a neat graphs prove that the area enclosed by the arcs OA , AB and the line OB is unit square units.
- (b) If N is a foot of the perpendicular from A to the axis of x , find by integration the volume obtained when the area enclosed by AN , NB and the arc AB is completely rotated about the x -axis, giving the answer correct to five significant figures.

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68. Evaluate the following integrals

(a) $\int_{2i}^{3i} (x^2 + 2x - \sinh 2x) dx$ (b) $\int_{3i}^5 \left(\frac{3}{2+3x} \right) dx$

69. Find the volume of solid of revolution formed by revolving the finite region bounded by the curve $y^2 = x - 1$ and $y = (x - 1)^2$ about the y -axis.

70. Find the volume of the solid of revolution generated by rotating the region in the first quadrant of the curve $y = (x - 1)(x - 3)^2$, $y = 0$ about y -axis.

71. Find the volume of the solid generated when the region bounded by the curve $y = x^3 - 3x + 3$, $x = 0$, $y = 0$ and $x = 2$ is rotated about the y -axis.

72. Find the volume of the solid obtained by rotating the region enclosed by the curve $y = x^2$ and $y = 4x - x^2$ about the line $x = 2$.

73. Consider the region enclosed by the curve $f(x) = x^3 + x$, $x = 2$ and the x -axis. Rotate this region about the y -axis and find the resulting volume.

74. Find the volume of the solid revolution generated when the region defined by the curve $y^2 = x^2(x + 1)$ about the y -axis.

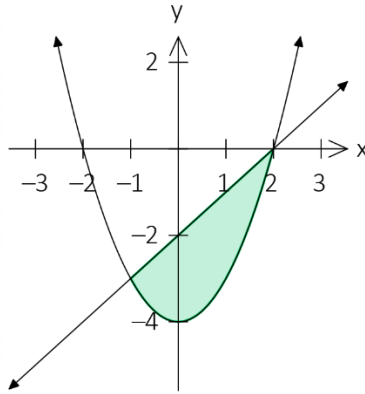
75. Find the volume of the solid generated by revolving the region bounded by $y = \frac{1}{1 + \sqrt{x - 2}}$, $y = 0$, $x = 2$ and $x = 6$ about the y -axis.

76. For the region bounded by $y = \frac{x}{\sqrt[3]{x + 1}}$, $y = 0$, $x = 0$ and $x = 7$. Find the volume of the solid generated by revolving the region about the x -axis.

77. Evaluate (a) $\int \cos(x) \ln(\sin x) dx$ (b) $\int \sin^{-1} \left(\frac{2x}{1 + x^2} \right) dx$

78. Show that $\int_{-1}^1 \frac{1}{\sqrt{6 + x - x^2}} dx = \sin^{-1} \left(\frac{6\sqrt{6} + 4}{25} \right)$

79. Find the area between the curves given in the figure below.



80. The slope of a curve at a point whose abscissa is x is $1 + \frac{x^2}{2}$ and the curve passes through the point $x = 1$ and $y = 0$. Find the equation of the curve and the area bounded by it, with the x -axis and the abscissa $x = 1$ and $x = 3$

81. Find the expression of $\int \left(\frac{dy}{dx} + 3x^2 + \sin(3x) \right) dx$

82. Show that $\int \frac{f'(x)}{f(x)} dx = \ln A f(x)$, hence show that if

$$\ln(f(x))^5 = \frac{1}{2} \int \frac{f'(x)}{f(x)} dx \text{ then } A = (f(x))^9$$

83. (a) Evaluate $\int_0^{\pi/2} (x \sin(x) \cos(x)) dx$

(b) Find $\int (\cos 2x \cos 4x \cos 6x) dx$

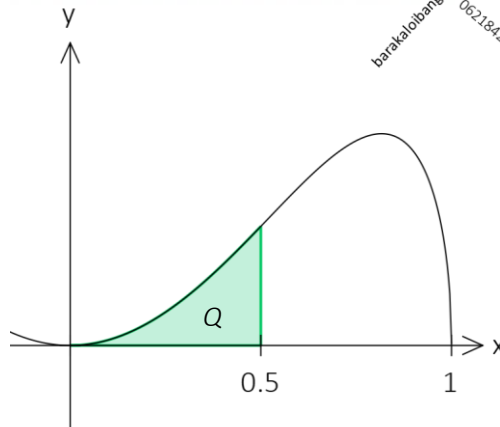
84. Show that $\int \frac{x}{a-bx} dx = -\frac{a}{b^2} \ln(a-bx) + \frac{1}{b^2} (a-bx) + A$ where a, b and A are real constants.

85. Find (a) $\int \frac{1}{x \ln x} dx$ (b) $\int \frac{1}{x \sqrt{x-1}} dx$ (c) $\int x \sqrt{x^4 - 9} dx$

86. Find (a) $\int \frac{x}{1+e^{-x^2}} dx$ (b) $\int \frac{\sin 4x}{2 + \cos 2x} dx$

87. Show that the volume represented by a solid of revolution of the curve $x^2 + y^2 = r^2$ from $x = -r$ to $x = r$ is a sphere.

88. Show that $\int_0^{\pi} \frac{x}{1 + \cos^2 x} dx = \frac{\pi}{2\sqrt{2}}$
89. If $\int_a^2 (x^2 + 3x - 4) dx = 4.5$ find the possible value(s) of a
90. Show that $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx = A - \frac{1}{2} \cos\left(\frac{\pi}{2} - 2x\right)$
91. Find $\int \frac{x^3 \tan^{-1} x^4}{1 + x^8} dx$
92. Show that $\int \tan^{-1} \sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}} dx = \frac{\pi}{4} x - \frac{1}{2} x^2 + A$
93. Show that for all real values of m , $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx = \frac{\pi}{4}$ hence integrate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$.
94. (a) The diagram below shows the shaded region Q enclosed between the curve $y = x^2 \sqrt{1 - x^2}$, the x -axis, and the line $x = 0.5$.



The area of this region is given by $A = \int_0^{0.5} x^2 \sqrt{1 - x^2} dx$, find the area Q

- (b) Find the exact volume generated when the region Q is rotated through four right angles about the x -axis.

95. Find (a) $\int \frac{x^4}{1+x^{10}} dx$ (b) $\int \cos^{-1}(\sin 3x) dx$ (c) $\int \frac{\operatorname{cosec}^2 x}{\sec^2 x} dx$
96. (a) Show that $\int \frac{5-8\sin x}{\cos^2 x} dx = 5 \tan x - 8 \sec x + A$
- (b) Show that $\int_a^b \frac{x^2-1}{x^2+1} dx = (b-a) - 2(\tan^{-1} a - \tan^{-1} b)$
97. (a) Verify that $\int \frac{\sin 2x}{1+\sin 2x} dx = 2 \sec 2x - 2 \tan 2x + 2x + A$
- (b) The marginal cost of a firm with fixed cost of 30 is given by $f(x) = 5 + 50x - 4x^2$, find the cost function.
- (c) The slope of the curve is given by $\frac{x^2+3x}{\sqrt{2x^3+9x^2+4}}$, if the curve passes through point $(0,1)$ find the equation of the curve.
- (d) Find the area bounded by the curve $y = \ln(x^2+3)$ with the x -axis from $x = -4$ to $x = 4$.
98. (a) Find $\int \frac{\cos x - \cos 2x}{1+2\cos x} dx$
- (b) Suppose $f'(x) = (x+1)(x-b)$, $f(0)=2$ and $f(1)=3$, find $f(x)$
- (c) Oil flows from the bottom of a storage tank at the rate of $r(t) = 150 - 5t$ litres per minutes, where $0 \leq t \leq 30$. Find the amount of oil that flows from the tank during the first 20 minutes.
- (d) A car accelerates along a straight line, so that its acceleration at time t is $a(t) = (t^2 - 3t) \text{ ms}^{-2}$. Find the displacement of the car during the time period $2 \leq t \leq 3$
99. (a) Use the concept of complex numbers or otherwise, find $\int (\sin^4 x + \cos^5 x) dx$
- (b) Find $\int \frac{2\cos x + 9\sin x}{3\cos x + \sin x} dx$
- (c) Find $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$
- (d) Determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 2$ and $y = x$ about line $y = 4$.
100. (a) Find the area inside the curve $y^2 = x^2 - x^4$
- (b) Using calculus show that the perimeter (arc length) of the circle with radius r is $2\pi r$.

- (c) The area defined by the inequalities $y \geq x^2 + 1$, $x \geq 0$, $y \leq 2$ is rotated completely about the y -axis. Find the volume of the solid generated.

101. Integrate the following:-

(a) $\int \frac{4x^3}{x^8 + 1} dx$

(b) $\int \frac{x^3}{x^8 - 1} dx$

(c) $\int \frac{x^2 - 1}{x^3 + x^2} dx$

(d) $\int \frac{x^3 - x}{x^6 + 3x^4 + 3x^2 + 1} dx$

102. Integrate the following:-

(a) $\int \frac{\tan^{-1}(x)}{1 + x^2} dx$

(b) $\int \frac{\arctan^2(x)}{x^2 + 1} dx$

(c) $\int \frac{\ln(x+1)}{x+1} dx$

(d) $\int \frac{(x+2)\ln(x+1)}{x-1} dx$

103. Show that $\int \frac{x^5 - x^3 + 1}{x^4 - x^3} dx = \frac{x^2}{2} + x + \frac{1}{2x^2} + \frac{1}{x} + \ln\left(\frac{x-1}{x}\right) + A$

104. Show that $\int \frac{x}{\sqrt{1-x^2}} dx = x - \sqrt{1-x^2} \sin^{-1} x + A$

105. Show that $\int_1^2 \frac{4x^2 + 4x - 17}{2x^2 + 5x - 3} dx = 2 + \ln\left(\frac{4}{45}\right)$

ADVANCED MATHEMATICS

INTEGRATION

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