QN: Integrate $\int \sqrt{\tan x}\, dx$

SOLUTION

Let

$I = \int \sqrt{\tan x}\, dx$=?

Use the u substitution as follows

Let $u = \sqrt{\tan x}$ (1)

Therefore

$$u^2 = \tan x$$

and upon differentiating;

$$2udu = \sec^2 x dx$$

$$\frac{2udu}{\sec^2 x} = dx$$

$$\frac{2udu}{1+u^4} = dx \quad (2)$$

Therefore

$$\int \sqrt{t\tan x}\, dx = \int u\left(\frac{2u}{1+u^4}\right) du$$

$$\int \frac{2u^2}{1+u^4} du$$

Note that

$$\frac{2u^2}{1+u^4} = \frac{2u^2}{(u^2-i)(u^2+i)}$$

Before going further, decompose the above function into partial fractions, i.e. let

$$\frac{2u^2}{1+u^4} = \frac{A}{(u^2-i)} + \frac{B}{(u^2+i)}$$

implying that

$$2u^2 = A(u^2+i) + B(u^2-i)$$

and upon equating the *coefficients having same powers of* u we obtain the equations

$$A + B = 2$$

and

$$Ai - Bi = 0 \text{ or in other words } -A + B = 0$$

which upon solving gives the solution $A = B = 1$

Therefore

$$\frac{2u^2}{1+u^4} = \frac{1}{(u^2 - i)} + \frac{1}{(u^2 + i)}$$

Upon multiplying by i to both the numerators and the denominators, of this equation, we obtain

$$\frac{2u^2}{1+u^4} = \frac{i}{(iu^2 + 1)} + \frac{i}{(iu^2 - 1)}$$

and eventually the expression

$$\frac{2u^2}{1+u^4} = \frac{i}{(1 + iu^2)} - \frac{i}{(1 - iu^2)}$$

is obtained after multiplying by -1 to both the numerator and the denominator of the term to the extreme right, which is $i/(iu^2 - 1)$

Therefore

$$\int \frac{2u^2}{1+u^4}\,du = \int \frac{i}{(1 + iu^2)}\,du - \int \frac{i}{(1 - iu^2)}\,du$$

For easy handling of the problem, let

$$\int \frac{2u^2}{1+u^4}\,du = I = I_i - I_{-i}$$

where

$$I_i = \int \frac{i}{(1 + iu^2)}\,du$$

and

$$I_{-i} = \int \frac{i}{(1 - iu^2)}\,du$$

Recall also from the identities (below) that

$$\int \frac{dv}{(1+v^2)} = \tan^{-1} v = -\frac{i}{2}\ln\left(\frac{1+iv}{1-iv}\right)$$

$$\int \frac{dw}{(1-w^2)} = \tanh^{-1} w = \frac{1}{2}\ln\left(\frac{1+w}{1-w}\right)$$

and in a more general form

$$\int \frac{a}{(1+u^2)}\,du = \sqrt{a}\tan^{-1}(u\sqrt{a}) = \frac{-i\sqrt{a}}{2}\ln\left(\frac{1+iu\sqrt{a}}{1-iu\sqrt{a}}\right)$$

$$\int \frac{a}{(1-u^2)}\,du = \sqrt{a}\tanh^{-1}(u\sqrt{a}) = \frac{\sqrt{a}}{2}\ln\left(\frac{1+u\sqrt{a}}{1-u\sqrt{a}}\right)$$

Using the above identities therefore, we can easily find the solutions to I_i and I_{-i} as follows:

$$I_i = \int \frac{i}{(1+iu^2)}\,du = \sqrt{i}\tan^{-1}(u\sqrt{i}) = \frac{-i\sqrt{i}}{2}\ln\left(\frac{1+iu\sqrt{i}}{1-iu\sqrt{i}}\right)$$

$$I_{-i} = \int \frac{i}{(1-iu^2)}\,du = \sqrt{i}\tanh^{-1}(u\sqrt{i}) = \frac{\sqrt{i}}{2}\ln\left(\frac{1+u\sqrt{i}}{1-u\sqrt{i}}\right)$$

Before going further, let us try to find the exact value of $\sqrt{i}$ so that we can substitute it into both of the expressions above

Let

$$a + ib = \sqrt{i}$$

Then

$$(a+ib)^2 = \left(\sqrt{i}\right)^2$$

$$a^2 + 2iab - b^2 = i$$

implying that

$$a^2 - b^2 = 0$$

$$2ab = 1$$

when we equate real and imaginary coefficients/ parts of the equation $a^2 + 2iab - b^2 = i$

Therefore

$$a^2 = b^2$$

meaning that

$$a = (b, -b)$$

However the solution $a = -b$ doesn't satisfy the second equation $2ab = 1$ for any real number b. Therefore $a = b$ is the only solution so that

$$a = b = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

Therefore the roots of i are

$$\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1 + i)$$

and

$$\sqrt{i} = -\frac{1}{\sqrt{2}}(1 + i)$$

Hence

$$\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1 + i)$$

Case 1:

Now back to both our integrands and with these two values of $\sqrt{i}$ beginning with the value of

$$\sqrt{i} = \frac{1}{\sqrt{2}}(1 + i)$$

Important review

Let a complex number z be given as $z = \cos\varphi + i\sin\varphi$. Then

$$\frac{dz}{d\varphi} = -\sin\varphi + i\cos\varphi$$

$$\frac{dz}{d\varphi} = i(\cos\varphi + i\sin\varphi)$$

$$\frac{dz}{d\varphi} = iz$$

$$\frac{dz}{z} = id\varphi$$

$$\int \frac{dz}{z} = \int id\varphi$$

$$\ln z = i\varphi + C;\ C \in R$$

$$z = e^{i\varphi + C}$$

Recall that $z = \cos\varphi + i\sin\varphi$ so that when $\varphi = 0^o$

$$z = \cos 0^o + i\sin 0^o$$
$$z = 1 + 0$$
$$z = 1$$

Hence $z = 1$when $\varphi = 0^o$

$$z = e^{i\varphi + C}$$
$$1 = e^{0 + C}$$
$$1 = e^{C}$$

Implying that $C = 0$ and hence $z = e^{i\varphi}$ or $z = re^{i\varphi} = e^{i\varphi}$ when $r = 1$ and when $r \neq 1$

$z = re^{i\varphi}$ or $z = r(\cos\varphi + i\sin\varphi) = x + iy$ in Cartesian coordinates where $x = r\cos\varphi$ and $y = r\sin\varphi$

Therefore from these polar and Cartesian coordinates' relationship we eventually deduce

$r^2 = x^2 + y^2$and $\tan\varphi = y/x$ or simply

$r = \sqrt{x^2 + y^2}$

$\varphi = \tan^{-1}\dfrac{y}{x}$

Hence, upon differentiating $z = re^{i\varphi}$

$\ln z = \ln\left(re^{i\varphi}\right)$

$\ln z = \ln r + i\varphi = \ln r + i(\varphi + 2n\pi)$ implying that

$$\ln z = \ln\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x}$$

$$\ln(x + iy) = \ln\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x}$$

since

$$r = \sqrt{x^2 + y^2}$$

and

$$\varphi = \tan^{-1}\frac{y}{x}$$

Then since

$$I_i = \int \frac{i}{(1+iu^2)}du = \sqrt{i}\tan^{-1}(u\sqrt{i}) = \frac{-i\sqrt{i}}{2}\ln\left(\frac{1+iu\sqrt{i}}{1-iu\sqrt{i}}\right)$$

$$I_{-i} = \int \frac{i}{(1-iu^2)}du = \sqrt{i}\tanh^{-1}(u\sqrt{i}) = \frac{\sqrt{i}}{2}\ln\left(\frac{1+u\sqrt{i}}{1-u\sqrt{i}}\right)$$

with

$$\sqrt{i} = [\frac{1}{\sqrt{2}}(1+i), -\frac{1}{\sqrt{2}}(1+i)]$$

Case 1:

$$\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$$

$$I_i = \int \frac{i}{(1+iu^2)}du = \frac{-i\sqrt{i}}{2}\ln\left(\frac{1+iu\sqrt{i}}{1-iu\sqrt{i}}\right)$$

$$I_i = \frac{-i\sqrt{i}}{2}\left(\ln(1+iu\sqrt{i}) - \ln(1-iu\sqrt{i})\right)$$

Also

$$I_{-i} = \int \frac{i}{(1-iu^2)}du = \frac{\sqrt{i}}{2}\ln\left(\frac{1+u\sqrt{i}}{1-u\sqrt{i}}\right)$$

$$I_{-i} = \frac{\sqrt{i}}{2}\left(\ln(1+u\sqrt{i}) - \ln(1-u\sqrt{i})\right)$$

Substituting the value of $\sqrt{i}$ in the expressions of I_i we get

$$I_i = \frac{-i\frac{1}{\sqrt{2}}(1+i)}{2}\left(\ln\left(1+iu\frac{1}{\sqrt{2}}(1+i)\right) - \ln\left(1-iu\frac{1}{\sqrt{2}}(1+i)\right)\right)$$

$$I_i = \frac{-i(1+i)}{2\sqrt{2}}\left(\ln\left(1+\frac{iu}{\sqrt{2}}(1+i)\right) - \ln\left(1-\frac{iu}{\sqrt{2}}(1+i)\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(1+\frac{iu-u}{\sqrt{2}}\right) - \ln\left(1-\frac{iu-u}{\sqrt{2}}\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(\frac{\sqrt{2}+iu-u}{\sqrt{2}}\right)-\ln\left(\frac{\sqrt{2}-iu+u}{\sqrt{2}}\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(\frac{\sqrt{2}-u+iu}{\sqrt{2}}\right)-\ln\left(\frac{\sqrt{2}+u-iu}{\sqrt{2}}\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\left(\ln\left(\sqrt{2}-u+iu\right)-\ln\sqrt{2}\right)-\left(\ln\left(\sqrt{2}+u-iu\right)-\ln\sqrt{2}\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(\sqrt{2}-u+iu\right)-\ln\sqrt{2}-\ln\left(\sqrt{2}+u-iu\right)+\ln\sqrt{2}\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(\sqrt{2}-u+iu\right)-\ln\left(\sqrt{2}+u-iu\right)\right)$$

Now using the identity

$$\ln\left(x+iy\right) = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$

with $x = \sqrt{2}-u$ and $y = u$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(\sqrt{2}-u+iu\right)-\ln\left(\sqrt{2}+u-iu\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(\left(\sqrt{2}-u\right)^2+(u)^2\right)^{\frac{1}{2}} + i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right.$$
$$\left.-\left(\ln\left(\left(\sqrt{2}+u\right)^2+(u)^2\right)^{\frac{1}{2}} + i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(2-2\sqrt{2}u+u^2+u^2\right)^{\frac{1}{2}} + i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right.$$
$$\left.-\ln\left(2+2\sqrt{2}u+u^2+u^2\right)^{\frac{1}{2}} - i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\ln\left(2-2\sqrt{2}u+2u^2\right)^{\frac{1}{2}} + i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right) - \ln\left(2+2\sqrt{2}u+2u^2\right)^{\frac{1}{2}} - i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)$$

$$I_i = \frac{-i+1}{2\sqrt{2}}\left(\frac{1}{2}\ln\left(2-2\sqrt{2}u+2u^2\right) + i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right) - \frac{1}{2}\ln\left(2+2\sqrt{2}u+2u^2\right) - i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)$$

$$I_i = \frac{-i+1}{4\sqrt{2}}\left(\ln\left(2-2\sqrt{2}u+2u^2\right) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right) - \ln\left(2+2\sqrt{2}u+2u^2\right) - 2i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)$$

$$I_i = \frac{-i+1}{4\sqrt{2}}\left(\ln 2\left(1-\sqrt{2}u+u^2\right) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right) - \ln 2\left(1+\sqrt{2}u+u^2\right) - 2i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)$$

$$I_i = \frac{-i+1}{4\sqrt{2}}\left(\ln 2 + \ln\left(1-\sqrt{2}u+u^2\right) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right) - \ln 2 - \ln\left(1+\sqrt{2}u+u^2\right) - 2i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)$$

$$I_i = \frac{-i+1}{4\sqrt{2}}\left(\ln\left(1-\sqrt{2}u+u^2\right) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right) - \ln\left(1+\sqrt{2}u+u^2\right) - 2i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)$$

$$I_i = \frac{-i+1(-1)}{4\sqrt{2}}\left(-\ln\left(u^2-\sqrt{2}u+1\right)-2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right.$$
$$\left.+\ln\left(u^2+\sqrt{2}u+1\right)+2i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right)$$

$$I_i = \frac{-1+i}{4\sqrt{2}}\left(\ln\left(u^2+\sqrt{2}u+1\right)+2i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right.$$
$$\left.-\ln\left(u^2-\sqrt{2}u+1\right)-2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)$$

$$I_i = \frac{-1+i}{4\sqrt{2}}\left(\ln\left(u^2+1+\sqrt{2}u\right)+2i\tan^{-1}\left(\frac{-u}{\sqrt{2}+u}\right)\right.$$
$$\left.-\ln\left(u^2+1-\sqrt{2}u\right)-2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)$$

$$\boldsymbol{I_i = \frac{-1+i}{4\sqrt{2}}\left(\ln\left(u^2+1+\sqrt{2}u\right)-2i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right)\right.}$$
$$\boldsymbol{\left.-\ln\left(u^2+1-\sqrt{2}u\right)-2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)}$$

Likewise

$$I_{-i} = \frac{\sqrt{i}}{2}\left(\ln\left(1+u\sqrt{i}\right)-\ln\left(1-u\sqrt{i}\right)\right)$$

$$I_{-i} = \frac{\frac{1}{\sqrt{2}}(1+i)}{2}\left(\ln\left(1+u\frac{1}{\sqrt{2}}(1+i)\right)-\ln\left(1-u\frac{1}{\sqrt{2}}(1+i)\right)\right)$$

$$I_{-i} = \frac{1+i}{2\sqrt{2}}\left(\ln\left(1+\frac{u}{\sqrt{2}}(1+i)\right)-\ln\left(1-\frac{u}{\sqrt{2}}(1+i)\right)\right)$$

$$I_{-i} = \frac{1+i}{2\sqrt{2}}\left(\ln\left(1+\frac{u}{\sqrt{2}}+\frac{iu}{\sqrt{2}}\right)-\ln\left(1-\frac{u}{\sqrt{2}}-\frac{iu}{\sqrt{2}}\right)\right)$$

$$I_{-i} = \frac{1+i}{2\sqrt{2}}\left(\ln\left(\frac{\sqrt{2}+u+iu}{\sqrt{2}}\right)-\ln\left(\frac{\sqrt{2}-u-iu}{\sqrt{2}}\right)\right)$$

$$I_{-i} = \frac{1+i}{2\sqrt{2}}\left(\ln\left(\sqrt{2}+u+iu\right)-\ln\sqrt{2}-\ln\left(\sqrt{2}-u-iu\right)+\ln\sqrt{2}\right)$$

$$I_{-i} = \frac{1+i}{2\sqrt{2}}\left(\ln\left(\sqrt{2}+u+iu\right)-\ln\left(\sqrt{2}-u-iu\right)\right)$$

$$I_{-i} = \frac{1+i}{2\sqrt{2}}\left(\ln\left(\left(\sqrt{2}+u\right)^{2}+(u)^{2}\right)^{\frac{1}{2}}+i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right)\right.$$
$$\left.-\left(\ln\left(\left(\sqrt{2}-u\right)^{2}+(u)^{2}\right)^{\frac{1}{2}}+i\tan^{-1}\left(\frac{-u}{\sqrt{2}-u}\right)\right)\right)$$

$$I_{-i} = \frac{1+i}{2\sqrt{2}}\left(\ln\left(2u^{2}+2+2\sqrt{2}u\right)^{\frac{1}{2}}+i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right)\right.$$
$$\left.-\left(\ln\left(2u^{2}+2-2\sqrt{2}u\right)^{\frac{1}{2}}+i\tan^{-1}\left(\frac{-u}{\sqrt{2}-u}\right)\right)\right)$$

$$I_{-i} = \frac{1+i}{4\sqrt{2}}\left(\ln 2+\ln\left(u^{2}+1+\sqrt{2}u\right)+2i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right)\right.$$
$$\left.-\left(\ln 2+\ln\left(u^{2}+1-\sqrt{2}u\right)+2i\tan^{-1}\left(\frac{-u}{\sqrt{2}-u}\right)\right)\right)$$

Therefore

$$\boldsymbol{I_{-i} = \frac{1+i}{4\sqrt{2}}\left(\ln\left(u^2 + 1 + \sqrt{2}u\right) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right)\right.}$$
$$\boldsymbol{\left. - \ln\left(u^2 + 1 - \sqrt{2}u\right) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)}$$

Also, from above we found that

$$I_i = \frac{-1+i}{4\sqrt{2}}\left(\ln\left(u^2 + 1 + \sqrt{2}u\right) - 2i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right)\right.$$
$$\left. - \ln\left(u^2 + 1 - \sqrt{2}u\right) - 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)$$

Also recall from above that

$$\int \frac{2u^2}{1+u^4}\,du = I = I_i - I_{-i}$$

where

$$I_i = \int \frac{i}{(1+iu^2)}\,du$$

and

$$I_{-i} = \int \frac{i}{(1-iu^2)}\,du$$

Therefore

$$I = I_i - I_{-i}$$

For easy algebraic manipulations, let

$$u^2 + 1 + \sqrt{2}u = A$$
$$u^2 + 1 - \sqrt{2}u = B$$
$$\frac{u}{\sqrt{2}+u} = C$$
$$\frac{u}{\sqrt{2}-u} = D$$

Therefore

$$I_i = \frac{-1+i}{4\sqrt{2}}\left(\ln\left(u^2+1+\sqrt{2}u\right) - 2i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right) - \ln\left(u^2+1-\sqrt{2}u\right) - 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)$$

$$I_i = \frac{-1+i}{4\sqrt{2}}(\ln A - 2i\tan^{-1}C - \ln B - 2i\tan^{-1}D)$$

and

$$I_{-i} = \frac{1+i}{4\sqrt{2}}\left(\ln\left(u^2+1+\sqrt{2}u\right) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}+u}\right) - \ln\left(u^2+1-\sqrt{2}u\right) + 2i\tan^{-1}\left(\frac{u}{\sqrt{2}-u}\right)\right)$$

$$I_{-i} = \frac{1+i}{4\sqrt{2}}(\ln A + 2i\tan^{-1}C - \ln B + 2i\tan^{-1}D)$$

Therefore

$$I = I_i - I_{-i}$$

$$I_i = \frac{-1+i}{4\sqrt{2}}(\ln A - 2i\tan^{-1}C - \ln B - 2i\tan^{-1}D)$$

$$I_{-i} = \frac{1+i}{4\sqrt{2}}(\ln A + 2i\tan^{-1}C - \ln B + 2i\tan^{-1}D)$$

$$I = I_i - I_{-i}$$

$$I = \frac{-1+i}{4\sqrt{2}}(\ln A - 2i\tan^{-1}C - \ln B - 2i\tan^{-1}D) - \left(\frac{1+i}{4\sqrt{2}}(\ln A + 2i\tan^{-1}C - \ln B + 2i\tan^{-1}D)\right)$$

$$I = \frac{1}{4\sqrt{2}}(\ln A(i-1) - 2i\tan^{-1}C(i-1) - \ln B(i-1) - 2i\tan^{-1}D(i-1))$$
$$- (\frac{1}{4\sqrt{2}}(\ln A(1+i) + 2i\tan^{-1}C(1+i) - \ln B(1+i) + 2i\tan^{-1}D(1+i)))$$

$$I = \frac{1}{4\sqrt{2}}((i\ln A - \ln A + 2\tan^{-1}C + 2i\tan^{-1}C - i\ln B + \ln B + 2\tan^{-1}D + 2i\tan^{-1}D)$$
$$- (\ln A + i\ln A + 2i\tan^{-1}C - 2\tan^{-1}C - \ln B - i\ln B + 2i\tan^{-1}D - 2\tan^{-1}D))$$

$$I = \frac{1}{4\sqrt{2}}(i\ln A - \ln A + 2\tan^{-1}C + 2i\tan^{-1}C - i\ln B + \ln B + 2\tan^{-1}D + 2i\tan^{-1}D - \ln A - i\ln A - 2i\tan^{-1}C + 2\tan^{-1}C + \ln B + i\ln B - 2i\tan^{-1}D + 2\tan^{-1}D)$$

$$I = \frac{1}{4\sqrt{2}}(i\ln A - \ln A + 2\tan^{-1}C + 2i\tan^{-1}C - i\ln B + \ln B + 2\tan^{-1}D + 2i\tan^{-1}D - \ln A - i\ln A - 2i\tan^{-1}C + 2\tan^{-1}C + \ln B + i\ln B - 2i\tan^{-1}D + 2\tan^{-1}D)$$

Note that all ***complex terms marked in red*** cancel out, and the remaining ones are:

$$I = \frac{1}{4\sqrt{2}}(-\ln A + 2\tan^{-1}C + \ln B + 2\tan^{-1}D - \ln A + 2\tan^{-1}C + \ln B + 2\tan^{-1}D)$$

$$I = \frac{1}{4\sqrt{2}}(-2\ln A + 4\tan^{-1}C + 2\ln B + 4\tan^{-1}D)$$

$$I = \frac{2}{4\sqrt{2}}(-\ln A + 2\tan^{-1}C + \ln B + 2\tan^{-1}D)$$

$$I = \frac{1}{2\sqrt{2}}\ln(\frac{B}{A}) + 2\tan^{-1}C + 2\tan^{-1}D$$

But

$$A = u^2 + 1 + \sqrt{2}u$$

$$B = u^2 + 1 - \sqrt{2}u$$

$$C = \frac{u}{\sqrt{2} + u}$$

$$D = \frac{u}{\sqrt{2} - u}$$

Therefore

$$I = \frac{1}{2\sqrt{2}} \ln\left(\frac{u^2 + 1 - \sqrt{2}u}{u^2 + 1 + \sqrt{2}u}\right) + 2\tan^{-1}\left(\frac{u}{\sqrt{2} + u}\right) + 2\tan^{-1}\left(\frac{u}{\sqrt{2} - u}\right)$$

$$I = \int \sqrt{tanx}\, dx = \int \frac{2u^2}{1 + u^4}\, du$$

$$\boldsymbol{I = \int \frac{2u^2}{1 + u^4}\, du}$$

$$\boldsymbol{= \frac{1}{2\sqrt{2}} \ln\left(\frac{u^2 + 1 - \sqrt{2}u}{u^2 + 1 + \sqrt{2}u}\right) + 2\tan^{-1}\left(\frac{u}{\sqrt{2} + u}\right) + 2\tan^{-1}\left(\frac{u}{\sqrt{2} - u}\right)}$$

where $u = \sqrt{\tan x}$