

ADVANCED MATHEMATICS

DIFFERENTIATION

9

$$\frac{d^n y}{dx^n} = \frac{d}{dx} \left(\frac{d^{n-1} y}{dx^{n-1}} \right)$$

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To my daughters Gracious and Grace

DIFFERENTIATION

Differentiation is the process of finding the gradient of different curves. The rate of change of y with respect to x is called **derivative** of y . The first derivative is denoted by $f'(x)$ or dy/dx , which is defined as

$$f'(x) = \frac{\text{change in } y}{\text{change in } x}$$

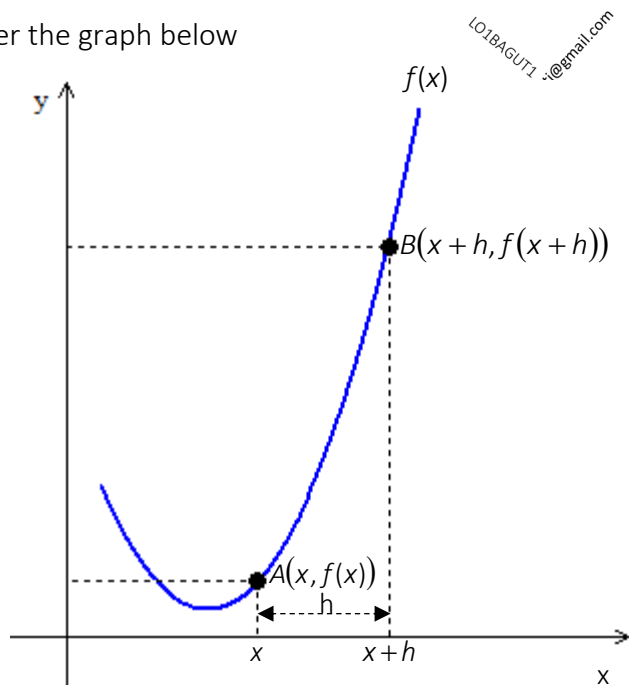
Chapter

9

9.1. Rules of differentiation

9.1.1. First principle of differentiation or definition of derivatives method.

Consider the graph below



B be close to each other, such that the horizontal distance between the points is negligible ($h \rightarrow 0$)

Then, the slope, $m = \frac{\Delta y}{\Delta x}$ thus $f'(x) = \frac{y_2 - y_1}{x_2 - x_1}$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{x+h-x} \right)$$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

This is the first principle of differentiation

Example 1

Use first principle of differentiation (definition of derivatives) to differentiate the following

(a) $f(x) = 2x + 3$ (b) $f(x) = x^2 - 3x + 5$ (c) $f(x) = 3x^2 + 4$

Solution

(a) $f(x) = 2x + 3$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left(\frac{2(x+h) + 3 - (2x+3)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{2x + 2h + 3 - 2x - 3}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{2h}{h} \right) = 2 \end{aligned}$$

Thus, $f'(x) = 2$ (slope of $y = 2x + 3$)

(b) $f(x) = x^2 - 3x + 5$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left(\frac{(x+h)^2 - 3(x+h) + 5 - (x^2 - 3x + 5)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{x^2 + 2xh + h^2 - 3x - 3h + 5 - x^2 + 3x - 5}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{2xh + h^2 - 3h}{h} \right) \\ &= \lim_{h \rightarrow 0} (2x + h - 3) = 2x - 3 \end{aligned}$$

Thus, $f'(x) = 2x - 3$

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$$(c) f(x) = 3x^2 + 4$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left(\frac{3(x+h)^2 + 4 - (3x^2 + 4)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{3x^2 + 6xh + 3h^2 - 3x^2 - 4}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{6xh + 3h^2}{h} \right) \\ &= \lim_{h \rightarrow 0} (6x + 3h) = 6x \end{aligned}$$

Thus, $f'(x) = 6x$

Example 2

Differentiate the following using first principle of differentiation

$$(a) f(x) = \frac{1}{x}$$

$$(b) f(x) = \sqrt{x}$$

$$(c) f(x) = \frac{1}{x^2 + 3}$$

$$(d) f(x) = \sqrt{x+2}$$

Solution

$$(a) f(x) = \frac{1}{x}$$

The first principle,

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{\frac{1}{x+h} - \frac{1}{x}}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{\frac{x - x - h}{x(x+h)}}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{-h}{x(x+h)} \right) \frac{1}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{-1}{x(x+h)} \right)$$

$$= -\frac{1}{x(x)} = -\frac{1}{x^2}$$

$$\text{Thus, } f'(x) = -\frac{1}{x^2}$$

$$(c) \quad f(x) = \sqrt{x}$$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right)$$

Rationalizing numerator

$$= \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{h}{h(\sqrt{x+h} + \sqrt{x})} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{1}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$\text{Thus, } f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$(b) \quad f(x) = \frac{1}{x^2 + 3}$$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{1}{(x+h)^2 + 3} - \frac{1}{x^2 + 3} \right) \frac{1}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{x^2 + 3 - ((x+h)^2 + 3)}{((x+h)^2 + 3)(x^2 + 3)} \right) \frac{1}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{x^2 + 3 - (x^2 + 2xh + h^2 + 3)}{((x+h)^2 + 3)(x^2 + 3)} \right) \frac{1}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{-2xh - h^2}{((x+h)^2 + 3)(x^2 + 3)} \right) \frac{1}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{-2x - h}{((x+h)^2 + 3)(x^2 + 3)} \right)$$

$$= \frac{-2x}{(x^2 + 3)(x^2 + 3)} = -\frac{2x}{(x^2 + 3)^2}$$

$$\text{Thus, } f'(x) = -\frac{2x}{(x^2 + 3)^2}$$

(d) $f(x) = \sqrt{x+2}$

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \times \frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{(x+h+2) - (x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})} \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} \right)$$

Therefore, $f'(x) = \frac{1}{2\sqrt{x+2}}$

EXERCISE 1

Use first principle to find dy/dx from the following

1. $f(x) = \frac{2}{\sqrt{x}}$

3. $f(x) = \frac{x+1}{x-1}$

5. $f(x) = (2x+5)^{-2}$

7. $f(x) = 4x^3 + (x+1)^2 - 2$

9. $f(x) = \frac{x^2 - 4}{x^2 + 3}$

11. $f(x) = \frac{21x+2}{x}$

2. $f(x) = 2\sqrt{x-3}$

4. $f(x) = \frac{3x+4}{x+3}$

6. $f(x) = \frac{1}{x} + x^2 - 3x$

8. $f(x) = \frac{4x}{\sqrt{x}}$

10. $f(x) = (x+1)^2$

12. $f(x) = 3x^3 + \sqrt{x}$

Power rule of differentiation

First principle become complicated as the degree of the relations become higher and higher. For the higher degree (fourth and above) expansion may be tiresome, unless the Pascal’s triangle is used to help in expansions.

Consider the following table with functions and their respective derivatives

Function	x^2	x^3	x^4	x^5	x^{-5}	x^{-n}	$x^{\frac{-1}{n}}$
Derivative	$2x$	$3x^2$	$4x^3$	$5x^4$	$-5x^{-6}$		

What did you notice from the table? How derivatives and their functions are related? Can you fill the last two columns of the table?

Compare your answers with the following nx^{n-1} and $-\frac{1}{n}x^{\frac{-1}{n}-1}$.

We have noticed that the derivative of $f(x)=x^n$ is $f'(x)=nx^{n-1}$, therefore the power rule of differentiation is if $y=x^n$ then

$\frac{dy}{dx} = nx^{n-1}$ where n is any real number.

Example 3

Differentiate $y = x^4 - x^3 + x^2 + 3$

Solution

Following the power rule, then

$y = x^4 - x^3 + x^2 + 3 \Rightarrow \frac{dy}{dx} = 4x^{4-1} - 3x^{3-1} + 2x^{2-1} + 0$

Therefore, $\frac{dy}{dx} = 4x^3 - 3x^2 + 2x$

➤ Note that the derivative of a constant, c is zero, thus

$$\frac{d}{dx}(c)=0 \text{ where } c \text{ is any number/constant.}$$

Proof

Let $f(x)=c$ where c is a constant.

$$f'(x)=\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \text{ thus } f'(x)=\lim_{h \rightarrow 0} \frac{c-c}{h}=\frac{0}{h}=0$$

The derivative of a constant function is zero.

Example 4

Find $\frac{dy}{dx}$ if $y=2x^3-4x^2+\frac{3}{x^2}+\frac{1}{x}+10$

Solution

Using power rule

$$y=2x^3-4x^2+\frac{3}{x^2}+\frac{1}{x}+10$$

$$\frac{dy}{dx}=\frac{d}{dx}(2x^3)-\frac{d}{dx}(4x^2)+\frac{d}{dx}\left(\frac{3}{x^2}\right)+\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{d}{dx}(10)$$

$$\frac{dy}{dx}=6x^2-8x-\frac{6}{x^3}-\frac{1}{x^2}$$

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EXERCISE 2

A)

1. Differentiate $y = \frac{x^3}{\sqrt[4]{x}} + 3x^{-4} - \sqrt[5]{2x^3} - 10x^{\frac{2}{3}}$
2. Use the concept of binomial expansion to prove that if $y = x^{-n}$ then $\frac{dy}{dx} = -nx^{-n-1}$
3. Use the principle of mathematical induction to show that $\frac{dy}{dx} = nx^{n-1}$ using the first principle.
4. Prove that the derivative of a constant is zero, let $y = cx^0$
5. Differentiate $f(x) = 1 + \frac{1}{x} - \frac{1}{x^3} - \frac{1}{3x^5}$
6. Differentiate $f(x) = \sqrt{x} + \frac{2}{\sqrt{x}} - 3\sqrt{x}$

B)

1. Differentiate by first principle

(a) $y = 2x^2 + 4x$

(f) $f(x) = \sqrt[3]{x^2} - 1$

(b) $f(x) = \frac{1}{x^2} + 3x$

(g) $f(x) = \frac{1}{\sqrt{x}}$

(c) $f(x) = x^3 - 3x^2 - 4$

(h) $f(x) = \sqrt{x-3}$

(d) $f(x) = 2x^2 - 5x + 3$

(i) $f(x) = \frac{3}{x^3}$

(e) $f(x) = \frac{2}{x^3}$

(j) Show that $x \frac{dy}{dx} = ny$ if $y = x^n$

2. Differentiate

(a) $y = 5x^8 - 32x^5 + 6x^{\frac{2}{3}} - 13x^3$

(b) $y = 2x^3 + 4x^{100} - 5x^{20}$

(c) $y = -3x^{10} + 5x^5 - 4x^2 - 7$

(d) $y = 3x^7 - \frac{2}{x^2} + 10x^3 + 20$

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Chain rule of differentiation

Chain rule is $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Chain rule is used to find the derivative of function of a function (composite functions). Suppose the function is $y = f(g(x))$

Let $u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$ and $y = f(u) \Rightarrow \frac{dy}{du} = f'(u)$

By chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = f'(x) \times g'(x)$$

But $u = g(x)$

$$\frac{dy}{dx} = f'(x)g'(x)$$

Example 5

Differentiate $y = (x^2 - 6)^2$

Solution

Given $y = (x^2 - 6)^2$

Let $u = x^2 - 6 \Rightarrow \frac{du}{dx} = 2x$ and $y = u^2 \Rightarrow \frac{dy}{du} = 2u$

Using $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = 2u(2x) = 4xu$$

But $u = x^2 - 6$

$$\frac{dy}{dx} = 4x(x^2 - 6)$$

Example 6

Find the derivative of $f(x) = (x^2 + 3x)^5$

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Solution

Given $f(x) = (x^2 + 3x)^5$

Let $u = x^2 + 3x$ then $y = u^5$, $\frac{du}{dx} = 2x + 3$ and $\frac{dy}{du} = 5u^4$

By chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = (5u^4)(2x + 3)$$

But $u = x^2 + 3x$

$$\frac{dy}{dx} = 5u^4(2x + 3)$$

Therefore, $\frac{dy}{dx} = 5(x^2 + 3x)^4(2x + 3)$

EXERCISE 3

Differentiate the following: -

1. $f(x) = \sqrt{2x^3 - 8x^{-2}}$

2. $y = \left(\frac{x}{\sqrt{x}} + 3x^{-3} - 4x^2 + 1 \right)^{-3}$

3. $y = \sqrt[7]{\frac{1}{4x^2} + 2x^3 - 3x^2 - 1}$

4. $y = (x^3 - 3x^{-2})^6$

5. $f(x) = \frac{x^3 - \sqrt{x} - 5x^2}{\sqrt[3]{2x}}$

6. If $f(x) = ax^{10} + bx^3$, $f(-1) = 3$ and $\frac{dy}{dx} = 5$ when $x = 1$ find a and b .

Product rule of differentiation

When a function is a product of two functions, the easiest way to get the derivative is by using the product rule. Suppose $y = uv$ where u and v are both functions of x . The changes in y affects the both functions u and v .

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

$$y + \Delta y = uv + u\Delta v + v\Delta u + \Delta u\Delta v$$

Then $\Delta u\Delta v \rightarrow 0$

$$\Delta y = u\Delta v + v\Delta u$$

Divide throughout by Δx

$$\frac{\Delta y}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x}$$

Apply limit $\lim_{h \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right) = u \left(\lim_{h \rightarrow 0} \frac{\Delta v}{\Delta x} \right) + v \left(\lim_{h \rightarrow 0} \frac{\Delta u}{\Delta x} \right)$

Therefore, the product rule; $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Example 7

Differentiate $y = (x^2 - 3)(x^3 + 7)$

Solution

Given $y = (x^2 - 3)(x^3 + 7)$

Let $u = x^2 - 3$ and $v = x^3 + 7$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\frac{dy}{dx} = (x^2 - 3)(3x^2) + (x^3 + 7)(2x)$$

Simplifying $\frac{dy}{dx} = 5x^4 - 9x^2 + 14x$

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Example 8

Find $\frac{dy}{dx}$ if $y = (x+2)^3(x^2+5x)$

Solution

Given $y = (x+2)^3(x^2+5x)$

Let $u = (x+2)^3$ and $v = (x^2+5x)$

Differentiate u by chain rule

$$\frac{dy}{dx} = (x+2)^3(2x+5) + 3(x^2+5x)(x+2)^2$$

$$\frac{dy}{dx} = (x+2)^2 \left[(x+2)(2x+5) + 3(x^2+5x) \right]$$

$$\frac{dy}{dx} = (x+2)^2 (5x^2 + 24x + 10)$$

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EXERCISE 4

1. Show that if $y = g^n$ then $\frac{dy}{dx} = ng^{n-1}g'$ where g is a function of x and n is any real number.
2. Derive the product rule of differentiation and use to answer question c) below
3. Differentiate with respect to x ;
 - (i) $y = (x+2)(x^2-5)^3$
 - (ii) $f(x) = (x^2+3x)^2(3x-2)^4$
 - (iii) $f(x) = (4x^3-2)^3(x^2+3)^2$
 - (iv) $y = (\sqrt{x}-2x)(x^3+3)^2$
4. Use product rule, find the derivative of $y = \frac{(2x+5)^3}{(x-2)^2}$
5. Use product rule show that if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$, hint use $y = (x^2)(x^{n-2})$
6. Differentiate $f(x) = (\sqrt{x^2-3x+4})^{-2}$

Quotient rule of differentiation

When a function is a quotient of two functions u and v where both are functions of x , to find its derivative we use the so-called quotient rule.

Form the product rule discussed above, $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Let $y = \frac{u}{v} = uv^{-1}$ using the product rule $\frac{dy}{dx} = u \left(-v^{-2} \right) \frac{dv}{dx} + v^{-1} \frac{du}{dx}$

$$\frac{dy}{dx} = -\frac{u}{v^2} \frac{dv}{dx} + \frac{1}{v} \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-u \frac{dv}{dx} + v \frac{du}{dx}}{v^2}$$

Hence, quotient rule, $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Example 9

Differentiate $y = \frac{x+2}{x^2-1}$

Solution

Let $u = x+2$ and $v = x^2-1$

Note that v is always the denominator function and u is a numerator function

$$\frac{dy}{dx} = \frac{(x^2-1)(1) - (x+2)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{x^2-1-2x^2-4x}{(x^2-1)^2}$$

Therefore, $\frac{dy}{dx} = -\frac{x^2+4x+1}{(x^2-1)^2}$

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Example 10

Differentiate $y = \frac{(x^2 + 2)^3}{(x^3 - 3)^2}$

Solution

Let $u = (x^2 + 2)^3$ and $v = (x^3 - 3)^2$

Differentiate both u and v by chain rule

$$y = \frac{(x^2 + 2)^3}{(x^3 - 3)^2}$$

$$\frac{dy}{dx} = \frac{6x(x^3 - 3)^2(x^2 + 2)^2 - 6x^3(x^2 + 2)^3(x^3 - 3)}{(x^3 - 3)^4}$$

$$\frac{dy}{dx} = \frac{6x(x^3 - 3)(x^2 + 2)^2[x^3 - 3 - x^2(x^2 + 2)]}{(x^3 - 3)^4}$$

Therefore,
$$\frac{dy}{dx} = \frac{6x(x^2 + 2)^2(x^3 - x^4 - 2x^2 - 3)}{(x^3 - 3)^3}$$

EXERCISE 5

1. Differentiate the following functions with respect to x

(a) $f(x) = \frac{x^2 + 1}{x^3 - 2}$

(b) $y = (x + 2)^2(x - 4)^{-1}$

(c) $y = (x^4 + 2x^3 - 1)^{10}$

(d) $y = (x^3 + 8)(x - 3)$

(e) $y = \frac{x + 1}{x - 1}$

(f) $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^3$

2. Differentiate the following with respect to x

$$(a) y = (x+2)(x-3)(x+5)$$

$$(c) y = ax(ax+b)^n$$

$$(b) y = \frac{3}{\sqrt{x^2-2}}, x \neq \sqrt{2}$$

$$(d) y = \frac{x^3+2x-5}{\sqrt{x}}, x \neq 0$$

3. Use $y = \frac{u}{v}$ where u and v are both functions of x , prove the quotient rule of differentiation.

4. Differentiate $y = \frac{1}{(x^2 - x + 3)^3}$

5. Differentiate $y = \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$

6. If $y = \frac{x-1}{x+1}$ find $\frac{dy}{dx}$

7. Differentiate $y = \frac{2x-1}{x^2-x+2}$

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Derivative of trigonometric functions and their inverses

Derivative of natural sine

Let $f(x) = \sin x$

By first principle of differentiation $\frac{dy}{dx} = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h \rightarrow 0} \left(\frac{\sin(x+h) - \sin x}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \right) \end{aligned}$$

$$\text{Limit, } \lim_{h \rightarrow 0} (\sin x (\cos h - 1)) = 0$$

$$= \lim_{h \rightarrow 0} \left(\frac{\cos x \sin h}{h} \right)$$

$$= \cos x \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right)$$

$$\text{Limit, } \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$$

$$= \cos x(1) = \cos x$$

$$\text{Therefore, } \frac{d}{dx}(\sin x) = \cos x$$

Derivative of the form, $y = \sin ax$

Let $y = \sin ax$, by using the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$u = ax, y = \sin u, \text{ then } \frac{du}{dx} = a \text{ and } \frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = (\cos u)a = a \cos u$$

$$\frac{dy}{dx} = a \cos ax$$

Generally, $y = \sin f(x)$ then $\frac{dy}{dx} = f'(x) \cos f(x)$

Example 11

Find the derivative of $y = \sin(2x^2 - 3x)$

Solution

Let $u = 2x^2 - 3x$ then $y = \sin u$

$$\frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = 4x - 3$$

Chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = (\cos u)(4x - 3)$$

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But $u = 2x^2 - 3x$

$$\frac{dy}{dx} = (4x - 3)\cos(2x^2 - 3x)$$

Example 12

Find the derivative of $y = \sin\left(\frac{2x}{x^2 - 3}\right)$

Solution

Given $y = \sin\left(\frac{2x}{x^2 - 3}\right)$

Let $u = \frac{2x}{x^2 - 3}$ (differentiate by quotient rule) then $y = \sin u$

$$\frac{du}{dx} = \frac{(x^2 - 3)(2) - 2x(2x)}{(x^2 - 3)^2}$$

$$\frac{du}{dx} = -\frac{2x^2 + 6}{(x^2 - 3)^2} \text{ and } \frac{dy}{du} = \cos u$$

$$\frac{dy}{dx} = -\frac{2x^2 + 6}{(x^2 - 3)^2} \cos u \text{ but } u = \frac{2x}{x^2 - 3}$$

$$\frac{dy}{dx} = -\frac{2x^2 + 6}{(x^2 - 3)^2} \cos\left(\frac{2x}{x^2 - 3}\right)$$

Derivative of natural cosine

Let $y = \cos x$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \left(\frac{\cos(x+h) - \cos x}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left(\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left(\frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \right) \\
 &= \lim_{h \rightarrow 0} (\cos x (\cos h - 1)) = 0 \\
 &= \lim_{h \rightarrow 0} \left(-\frac{\sin x \sin h}{h} \right)
 \end{aligned}$$

But $\lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) = 1$

$$= -\sin x \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) = -\sin x$$

Therefore, $\frac{d}{dx}(\cos x) = -\sin x$

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Derivative of the form, $y = \cos(bx)$

If $y = \cos bx$, let $u = bx \Rightarrow \frac{du}{dx} = b$ and $y = \cos u \Rightarrow \frac{dy}{du} = -\sin u$

Chain rule $\frac{dy}{dx} = (-\sin u)(b)$

Hence, $\frac{d}{dx}(\cos bx) = -b \sin bx$

Generally, $\frac{dy}{dx} = f'(x) \cos f(x)$

Example 13

Find the derivative of $y = \cos(3x^2)$

Solution

$$\text{Let } u = 3x^2 \Rightarrow \frac{du}{dx} = 6x \text{ then } y = \cos u \Rightarrow \frac{dy}{du} = -\sin u$$

$$\frac{dy}{dx} = (-\sin u)(6x) \text{ but } u = 3x^2$$

$$\frac{dy}{dx} = -6x \sin 3x^2$$

Example 14

Find the derivative of $y = \cos(\sin x)$

Solution

$$\text{Let } u = \sin x \text{ then } \frac{du}{dx} = \cos x \text{ and } y = \cos u \text{ then } \frac{dy}{du} = -\sin u$$

$$\frac{dy}{dx} = -\sin u(\cos x) \text{ but } u = \sin x$$

$$\text{Therefore } \frac{dy}{dx} = -\cos x \sin(\cos x)$$

Derivative of natural tangent

$$\text{Let } y = \tan x = \frac{\sin x}{\cos x} \text{ using the quotient rule}$$

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$$\frac{d}{dx}(\tan x) = \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$\frac{d}{dx}(\tan x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Derivative of the form, $y = \tan f(x)$

If $y = \tan(ax + b)$

Let $u = ax + b$ and $y = \tan u$, therefore, $\frac{du}{dx} = a$ and $\frac{dy}{du} = \sec^2 u$

$$\frac{dy}{dx} = (\sec^2 u)(a) \text{ but } u = ax + b \text{ therefore } \frac{dy}{dx} = a \sec^2(ax + b)$$

Generally, $y = \tan f(x) \Rightarrow \frac{dy}{dx} = f'(x) \sec^2 f(x)$

Example 15

Find the derivative $y = \tan 2x^3$

Solution

Let $u = 2x^3$ then $y = \tan u$

$$\frac{du}{dx} = 6x^2 \text{ and } \frac{dy}{du} = \sec^2 u$$

$$\frac{dy}{dx} = 6x^2 \sec^2 2x^3$$

Example 16

Differentiate $y = \tan(\sin 3x)$

Solution

Given $y = \tan(\sin 3x)$

Let $u = \sin 3x$ and $y = \tan u$ then

$$\frac{du}{dx} = 3 \cos 3x \text{ and } \frac{dy}{du} = \sec^2 u$$

$$\text{Chain rule } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \sec^2 u \times 3 \cos 3x$$

$$\text{Therefore, } \frac{dy}{dx} = 3 \sec^2(\sin 3x) \cos 3x$$

Example 17

Differentiate $y = \sin 4x \cos 2x$

Solution

Given $y = \sin 4x \cos 2x$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{dy}{dx} = (\sin 4x)(-2 \sin 2x)$$

$$+ (\cos 2x)(4 \cos 4x)$$

Therefore,

$$\frac{dy}{dx} = -2 \sin 4x \sin 2x + 4 \cos 2x \cos 4x$$

Example 19

Differentiate $y = x \tan 2x$

Solution

Given $y = x \tan 2x$

Using product rule $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\frac{dy}{dx} = (\tan 2x) + (x)(2 \sec^2 2x)$$

$$\frac{dy}{dx} = \tan 2x + 2x \sec^2 2x$$

Example 18

Differentiate $y = \tan 2x \cos x$

Solution

Given $y = \tan 2x \cos x$

$$\frac{dy}{dx} = (2 \sec^2 2x)(\cos x) + (-\sin x)(\tan 2x)$$

Hence,

$$\frac{dy}{dx} = 2 \sec^2 2x \cos x - \sin x \tan 2x$$

Example 20

Differentiate $y = \sqrt{\tan 3x}$

Solution

Given $y = \sqrt{\tan 3x}$

$$y^2 = \tan 3x \Rightarrow 2y \frac{dy}{dx} = 3 \sec^2 3x$$

$$\frac{dy}{dx} = \frac{3 \sec^2 3x}{2y}$$

$$\text{Hence, } \frac{d}{dx}(\sqrt{\tan 3x}) = \frac{3 \sec^2 3x}{2\sqrt{\tan 3x}}$$

Example 21

Differentiate $y = \cos 3x - x^2 \sin x$

Solution

Given $y = \cos 3x - x^2 \sin x$

$$\frac{dy}{dx} = (-3 \sin 3x) - (2x) \sin x - x^2 (\cos x)$$

$$\frac{dy}{dx} = -3 \sin 3x - 2x \sin x - x^2 \cos x$$

Derivative of natural cosecant

Let $y = \operatorname{cosec} x = \frac{1}{\sin x}$

$$\frac{dy}{dx} = \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x}$$

$$\frac{dy}{dx} = -\frac{\cos x}{\sin^2 x} = -\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right)$$

Therefore, $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Derivative of the form $y = \operatorname{cosec} f(x)$

Generally, $y = \operatorname{cosec} f(x) \Rightarrow \frac{dy}{dx} = -f'(x) \operatorname{cosec} f(x) \cot f(x)$

Example 22

Differentiate $y = \operatorname{cosec}(ax + b)$

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Solution

Given $y = \operatorname{cosec}(ax + b)$

Let $u = (ax + b)$ then $y = \operatorname{cosec} u$

$$\frac{du}{dx} = a \text{ and } \frac{dy}{du} = -\operatorname{cosec} u \cot u$$

Chain rule $\frac{dy}{dx} = -a \operatorname{cosec} u \cot u$

$$\frac{dy}{dx} = -a \operatorname{cosec}(ax + b) \cot(ax + b)$$

Derivative of natural secant

Let $y = \sec x = \frac{1}{\cos x}$

By quotient rule, $\frac{dy}{dx} = \frac{\sin x}{\cos^2 x} = \left(\frac{\sin x}{\cos x} \right) \left(\frac{1}{\cos x} \right)$

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Therefore, $\frac{d}{dx}(\sec x) = \sec x \tan x$

Derivative of the form $y = \sec f(x)$

Generally, $y = \sec f(x) \Rightarrow \frac{dy}{dx} = f'(x) \sec f(x) \tan f(x)$

Example 23

Differentiate $y = \sec(\cos 2x)$

Solution

Given $y = \sec(\cos 2x)$

Using the general conclusion above

$$\frac{dy}{dx} = -2 \sin 2x \sec(\cos 2x) \tan(\cos 2x)$$

Derivative of natural cotangent

By quotient rule

$$y = \cot x = \frac{1}{\tan x} \text{ thus}$$

$$\frac{dy}{dx} = -\frac{\sec^2 x}{\tan^2 x} = -\frac{1}{\cos^2 x} \times \frac{\cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

$$\text{Therefore, } \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\text{Derivative of the form } \frac{d}{dx}(\cot f(x)) = -f'(x) \operatorname{cosec}^2 f(x)$$

Example 24

Differentiate $y = x^5 \cot 2x$

Solution

Given $y = x^5 \cot 2x$

$$\frac{dy}{dx} = x^5(-\operatorname{cosec}^2 2x) + 5x^4 \cot 2x \text{ thus } \frac{dy}{dx} = 5x^4 \cot 2x - x^5 \operatorname{cosec}^2 2x$$

EXERCISE 6

1. Differentiate the following with respect to x .

(a) $y = \sin(2x+1)$

(b) $y = \tan(\sin x^2)$

(c) $y = \frac{1 + \sin x}{1 - \sin x}$

(d) $y = \sqrt{\frac{1 + \cos 2x}{1 - \cos 2x}}$

(e) $y = \sqrt{\frac{1 + \cos 4x}{2}}$

(f) $y = \sqrt{\frac{1 + \cos 2x}{2}}$

2. Differentiate with respect to x .

(a) $y = \sqrt{\sin \sqrt{x}}$

(b) $t = 2\sqrt{\cot x^2}$

(c) If $y = \sqrt{a^2 - x^2}$, show that $y \frac{dy}{dx} = -x$

(d) $y = \sec^2(x^2 - 1)$

3. Differentiate

(a) $y = \frac{\cos^2 x}{1 + \sin x}$

(b) If $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$ show that $\frac{dy}{dx} = \frac{1}{2y - 1}$

(c) $y = \cos(\sin x^2)$

(d) $y = \sec(\operatorname{cosec} \sqrt{x^2 - 1})$

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9.2. Derivative of trigonometric inverses

Derivative of $y = \sin^{-1}(x)$

$y = \sin^{-1}(x)$ this can be written as $x = \sin y$

$$\cos y \frac{dy}{dx} = 1 \text{ thus } \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} \text{ thus } \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$

Derivative of $y = \cos^{-1}(x)$

Expressing as $\cos y = x \Rightarrow -\sin y \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = -\frac{1}{\sin y} \quad \text{thus} \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

Derivative of $y = \tan^{-1}(x)$

This can be written as $\tan y = x \Rightarrow \sec^2 y \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \quad \text{thus} \quad \frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Derivative of $y = \sec^{-1}(x)$

$$\sec y = x \Rightarrow \sec y \tan y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \quad \text{thus} \quad \frac{dy}{dx} = \frac{1}{\sec y \sqrt{\sec^2 y - 1}}$$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

Derivative of $y = \operatorname{cosec}^{-1}(x)$

$$\operatorname{cosec} y = x \Rightarrow -\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y} \quad \text{thus} \quad \frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \sqrt{\operatorname{cosec}^2 y - 1}}$$

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$$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

Derivative of $y = \cot^{-1}(x)$

$$\cot y = x \Rightarrow -\operatorname{cosec}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y} \text{ thus } \frac{dy}{dx} = -\frac{1}{\cot^2 y + 1}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{x^2 + 1}$$

Example 25

Differentiate the following

(a) $y = \cos^{-1}(2x-1)$

(c) $y = \sec^{-1}(x^2+2)$

(b) $y = \tan^{-1}(x^2)$

(d) $y = \cot^{-1}(\cos x)$

Solution

(a) $y = \cos^{-1}(2x-1)$

$$\cos y = 2x-1$$

$$-\sin y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = -\frac{2}{\sin y} \text{ thus } \frac{dy}{dx} = -\frac{2}{\sqrt{1-\cos^2 y}}$$

$$\frac{dy}{dx} = -\frac{2}{\sqrt{1-(2x-1)^2}}$$

$$\frac{dy}{dx} = -\frac{2}{\sqrt{4x-4x^2}} = -\frac{1}{\sqrt{x-x^2}}$$

$$(b) \ y = \tan^{-1}(x^2)$$

$$\text{Then } \tan y = x^2$$

$$\sec^2 y \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{\sec^2 y} \text{ thus } \frac{dy}{dx} = \frac{2x}{1 + \tan^2 y}$$

$$\frac{dy}{dx} = \frac{2x}{1 + x^4}$$

$$(c) \ y = \sec^{-1}(x^2 + 2)$$

$$\sec y = x^2 + 2$$

$$\sec y \tan y \frac{dy}{dx} = 2x \text{ thus } \frac{dy}{dx} = \frac{2x}{\sec y \tan y}$$

$$\frac{dy}{dx} = \frac{2x}{\sec y \sqrt{\sec^2 y - 1}} \text{ thus } \frac{dy}{dx} = \frac{2x}{(x^2 + 2) \sqrt{(x^2 + 2)^2 - 1}}$$

$$\frac{dy}{dx} = \frac{2x}{(x^2 + 2) \sqrt{x^4 + 4x^2 + 3}}$$

$$(d) \ y = \cot^{-1}(\cos x)$$

$$\cot y = \cos x$$

$$-\operatorname{cosec}^2 y \frac{dy}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{\sin x}{\operatorname{cosec}^2 y} \text{ thus } \frac{dy}{dx} = \frac{\sin x}{\cot^2 y + 1}$$

$$\frac{dy}{dx} = \frac{\sin x}{\cos^2 x + 1}$$

Example 26

Show that if $\sin y = \cos x$ then $\frac{dy}{dx} = -1$

Solution

$$\cos y \frac{dy}{dx} = -\sin x \text{ thus } \frac{dy}{dx} = -\frac{\sin x}{\cos y}$$

$$\frac{dy}{dx} = -\frac{\sin x}{\sqrt{1 - \sin^2 y}} \text{ thus } \frac{dy}{dx} = -\frac{\sin x}{\sqrt{1 - \cos^2 x}}$$

$$\frac{dy}{dx} = -\frac{\sin x}{\sin x} = -1$$

Hence, $\frac{dy}{dx} = -1$

Derivative of the form $y = \sin^n(x)$

Let $u = \sin x$ then $y = u^n$ by chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{du}{dx} = \cos x \text{ and } \frac{dy}{du} = nu^{n-1}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \text{ thus } \frac{dy}{dx} = nu^{n-1} \cos x \text{ but } u = \sin x$$

$$\frac{dy}{dx} = n \sin^{n-1} x \cos x$$

Example 27

Show that $\sqrt{1-x^2} \frac{dy}{dx} = 3$ if $y = \sin^{-1}(3x-4x^3)$

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Solution

$$\sin y = 3x - 4x^2$$

$$\cos y \frac{dy}{dx} = 3 - 12x^2 \quad \text{thus} \quad \frac{dy}{dx} = \frac{3 - 12x^2}{\cos y}$$

$$\frac{dy}{dx} = \frac{3 - 12x^2}{\sqrt{1 - \sin^2 y}} \quad \text{thus} \quad \frac{dy}{dx} = \frac{3 - 12x^2}{\sqrt{1 - (3x - 4x^2)^2}}$$

$$\frac{dy}{dx} = \frac{3 - 12x^2}{\sqrt{1 - 9x^2 + 24x^4 - 16x^6}}$$

Factorizing the expression $1 - 9x^2 + 24x^4 - 16x^6 = (1 - x^2)(1 - 4x^2)^2$

$$\frac{dy}{dx} = \frac{3 - 12x^2}{(1 - 4x^2)\sqrt{1 - x^2}}$$

$$\frac{dy}{dx} = \frac{3(1 - 4x^2)}{(1 - 4x^2)\sqrt{1 - x^2}} \quad \text{thus} \quad \frac{dy}{dx} = \frac{3}{\sqrt{1 - x^2}}$$

Hence shown that $\sqrt{1 - x^2} \frac{dy}{dx} = 3$

Using trigonometric concept, can be a simple way to show this, let consider the identity $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$, this looks closely related to the identity $\sin y = 3x - 4x^2$. From $\sin y = 3x - 4x^2$ let $x = \sin \theta$, the identity become $\sin y = 3\sin \theta - 4\sin^3 \theta$, $\sin y = \sin 3\theta$ then $y = 3\theta$

Differentiate $y = 3\theta$ with respect to θ , $\frac{dy}{d\theta} = 3$ and $x = \sin \theta$,

$$\frac{dx}{d\theta} = \cos \theta \text{ thus by chain rule, } \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{3}{\cos \theta} \text{ thus } \frac{dy}{dx} = \frac{3}{\sqrt{1 - \sin^2 \theta}}$$

$$\sqrt{1 - x^2} \frac{dy}{dx} = 3. \text{ Hence shown}$$

Example 28

Show that $\frac{dy}{dx} = \sec^4 x$ if $y = \tan x + \frac{1}{3} \tan^3 x$

Solution

$$y = \tan x + \frac{1}{3} \tan^3 x$$

$$y = \tan x + \frac{1}{3} \tan^3 x \text{ thus}$$

$$\frac{dy}{dx} = \sec^2 x + \frac{1}{3} (3 \tan^2 x \sec^2 x)$$

$$\frac{dy}{dx} = \sec^2 x + \tan^2 x \sec^2 x$$

$$\frac{dy}{dx} = \sec^2 x (1 + \tan^2 x)$$

$$\text{Hence, } \frac{dy}{dx} = \sec^4 x$$

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EXERCISE 7

1. Show that $\sin(x) \frac{dy}{dx} + y = 0$ if $y = \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}$
2. If $\sin^2 y = \frac{1-\cos x}{2}$ show that $\frac{dy}{dx} = \frac{1}{2}$
3. If $\tan x = \frac{3t-t^3}{1-3t^2}$ show that $(1+t^2) \frac{dx}{dt} = 3$
4. Show that if $y = \frac{1}{\sqrt{1-x^2}}$ then $(1-x^2)^{3/2} \frac{dy}{dx} - x = 0$
5. Show that $\frac{dy}{dx} = 2 \sin\left(\frac{5\pi}{6} - x\right)$ if $y = \sin x + \sqrt{3} \cos x$
6. Prove that $\sqrt{1-x^2} \frac{dy}{dx} + x = \sqrt{1-x^2}$ if $\sin y = \left(\frac{x + \sqrt{1-x^2}}{\sqrt{2}} \right)$
7. Show that $\frac{dx}{dy} = -\sqrt{1-x^2}$ if $\cos y = \left(\frac{x + \sqrt{1-x^2}}{\sqrt{2}} \right)$
8. Let $x = \sin \theta$ or otherwise and $y = \sin^2 \left(\cot^{-1} \sqrt{\frac{1+x}{1-x}} \right)$, show that $\frac{dx}{dy} = -2$
9. Show that if $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$ then $\frac{dy}{dx} = \frac{1}{1-\sin x}$
10. If $\tan y = \frac{2x}{1-x^2}$ prove that $\frac{dy}{dx} = \frac{2}{1+x^2}$
11. Prove that $(1-x^2) \frac{dy}{dx} = xy + 1$ if $x = \sin^{-1} \left(y \sqrt{1-x^2} \right)$
12. Differentiate with respect to x , $y = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$

9.3. The derivatives of logarithmic functions

There are two bases, which are used commonly than other; and deserve special mention. These are

- (a) Base e logarithms (Natural logarithm)
- (b) Base 10 logarithms (Common logarithm)

The base e is called the exponential constant and has a value approximately equal to 2.718. Base e is used because this constant occurs frequently in the mathematical modelling of many physical, biological and economic applications. Such logarithms are also called Napierian or natural logarithms and abbreviated as **ln**, thus $\ln x = \log_e x$

The base 10 logarithm is referred as **common** logarithm and is denoted as $\log$, thus $\log 5 = \log_{10} 5$

9.3.1. Natural logarithm (ln)

Derivative of natural logarithm, $f(x)=\ln(x)$, for all $x > 0$

Using the first principle of differentiation

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\ \frac{dy}{dx} &= \lim_{h \rightarrow 0} \left(\frac{\ln(x+h) - \ln x}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \ln \left(1 + \frac{h}{x} \right) \\ &= \lim_{h \rightarrow 0} \frac{x}{xh} \ln \left(1 + \frac{h}{x} \right) \text{ thus } \left(\frac{1}{x} \right) \lim_{h \rightarrow 0} \ln \left(1 + \frac{h}{x} \right)^{\frac{x}{h}}\end{aligned}$$

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Consider $\lim_{\substack{h \rightarrow 0 \\ x \rightarrow \infty}} \left(1 + \frac{h}{x}\right)^{\frac{x}{h}} \rightarrow e$ thus $\left(\frac{1}{x}\right) \ln e$

Therefore, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Derivative of the form of $f(x) = \ln f(x)$

Let $u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$ and $y = \ln u \Rightarrow \frac{dy}{du} = \frac{1}{u}$ then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times f'(x) = \frac{f'(x)}{f(x)}$$

Therefore $\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$

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Example 29

Differentiate $y = \ln(x^2 - 3x)$

Solution

$$y = \ln(x^2 - 3x)$$

$$u = x^2 - 3x \Rightarrow \frac{du}{dx} = 2x - 3$$

$$y = \ln u \Rightarrow \frac{dy}{du} = \frac{1}{u} \text{ thus } \frac{dy}{dx} = \frac{1}{u}(2x - 3)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \text{ thus } \frac{dy}{dx} = \frac{2x - 3}{x^2 - 3x}$$

Example 30

Differentiate $y = \ln(\ln x^2)$

Solution

Given $y = \ln(\ln x^2)$

$$u = \ln x^2 \Rightarrow \frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x}$$

$$y = \ln u \Rightarrow \frac{dy}{du} = \frac{1}{u} \text{ thus } \frac{dy}{dx} = \frac{1}{u} \times \frac{2}{x}$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{2}{x \ln x^2}$$

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Example 31

Differentiate $y = \ln(\cos x^2)$

Solution

Given $y = \ln(\cos x^2)$

$$u = \cos x^2 \Rightarrow \frac{du}{dx} = -2x \sin x^2 \text{ thus } y = \ln u \Rightarrow \frac{dy}{du} = \frac{1}{u}$$

$$\text{Chain rule } \frac{dy}{dx} = -2x \sin x^2 \left(\frac{1}{u} \right) = -2x \frac{\sin x^2}{\cos x^2} = -2x \tan x^2$$

9.3.2. Derivative of common logarithm, $f(x) = \log_{10} M$, for $M > 0$

To differentiate the common logarithm, we need first to express the common logarithm into the natural logarithm form.

Derivative of the form of $y = \log_{10} f(x)$

Let $y = \log_{10} f(x)$

To exponential form $10^y = f(x)$

Apply the natural logarithm to both sides $\ln 10^y = \ln f(x)$

Make y the subject, $y \ln 10 = \ln f(x)$ as $y = \frac{1}{\ln 10} \ln f(x)$

Differentiate, $\frac{dy}{dx} = \frac{1}{\ln 10} \left(\frac{f'(x)}{f(x)} \right)$

Note that, using this method of changing common logarithm to natural logarithm we can differentiate any logarithm of any base.

Example 32

Differentiate $y = \log(x^2 - 2)$

Solution

$y = \log(x^2 - 2)$ in to exponential form $10^y = x^2 - 2$

$$y \ln 10 = \ln(x^2 - 2) \Rightarrow \ln 10 \frac{dy}{dx} = \frac{2x}{x^2 - 2}$$

$$\frac{dy}{dx} = \frac{1}{\ln 10} \left(\frac{2x}{x^2 - 2} \right)$$

Example 33

Find $\frac{dy}{dx}$ if $y = \log x^{x^3}$

Solution

$$y = \log x^{x^3} \Rightarrow 10^y = x^{x^3}$$

$$y \ln 10 = \ln x^{x^3} \text{ thus } y \ln 10 = x^3 \ln x$$

$$\ln 10 \frac{dy}{dx} = 3x^2 \ln x + x^3 \left(\frac{1}{x} \right)$$

$$\frac{dy}{dx} = \frac{1}{\ln 10} (3x^2 \ln x + x^2) \text{ thus } \frac{dy}{dx} = \frac{x^2}{\ln 10} (3 \ln x + 1)$$

EXERCISE 8

Differentiate the following, hint use natural logarithm

1. $y = \log(\ln x^2)$

2. $f(x) = \ln(\log x^2)$

3. $y = \ln\left(\frac{x^3 + 3x}{x - 2}\right)$

4. $f(x) = (2x^2 - 3)^{10} (3x^3 + 2)^4$

5. $f(x) = \frac{(x^3 - 9)^9}{(x^2 + 4)^5}$

6. $y = \cos^5(2x) \sin^3(3x^2)$

7. $f(x) = \tan(x) \cos^2(2x)$

8. $f(x) = 2 \cos(x^3) \sin^3(x)$

9. $y = \frac{\cos^3(3x)}{3 \sin^2(4x)}$

10. $y = \ln \frac{(x - 2x^2)}{(x^3 + 4)^5}$

11. $f(x) = \tan^3(x) \sin(2x^3)$

$$12. f(x) = \left(\frac{x+1}{x^2-3} \right)^5$$

$$13. f(x) = \sin(x^2 + 2x) \cos(4 - x^3)^5 \quad 14. y = \cos(x^\circ) \sin^3(x^\circ)$$

9.4. Derivative of exponential function

In this part, we will consider exponential functions of type $f(x) = e^x$ and $f(x) = a^x$ where e is a constant approximated to 2.7183 (to four decimal places) and a is a non-zero constant.

Derivative of $y = e^x$

By first principle, $\frac{dy}{dx} = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \left(\frac{e^{x+h} - e^x}{h} \right)$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \left(\frac{e^x e^h - e^x}{h} \right)$$

$$\frac{dy}{dx} = e^x \lim_{h \rightarrow 0} \left(\frac{e^h - 1}{h} \right)$$

But $\lim_{h \rightarrow 0} (e^h - 1) \rightarrow h$ thus $\frac{d}{dx}(e^x) = e^x$

Derivative of the form $y = e^{f(x)}$

Let $u = f(x) \Rightarrow \frac{du}{dx} = f'(x)$ and $y = e^u \Rightarrow \frac{dy}{du} = e^u$

Chain rule $\frac{dy}{dx} = f'(x)e^u$

Therefore, $\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$

Example 34

Differentiate $y = e^{\sin x}$

Solution

Let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x$ and $y = e^u \Rightarrow \frac{dy}{du} = e^u$

$$\frac{dy}{dx} = e^u (\cos x)$$

Hence $\frac{dy}{dx} = (\cos x)e^{\sin x}$

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Example 35

Find the derivative of $y = e^{e^{x^2}}$

Solution

Let $u = e^{x^2} \Rightarrow \frac{du}{dx} = 2xe^{x^2}$ and $y = e^u \Rightarrow \frac{dy}{du} = e^u$

$$\frac{dy}{dx} = e^u \times (2xe^{x^2})$$

Hence, $\frac{dy}{dx} = 2xe^{(x^2 + e^{x^2})}$

Derivative of $y = a^x$

Introduce natural logarithm both sides

$$\ln y = \ln a^x$$

$$\ln y = x \ln a$$

Differentiate $\frac{1}{y} \frac{dy}{dx} = \ln a$

$$\frac{dy}{dx} = y \ln a$$

Therefore, $\frac{d}{dx}(a^x) = a^x \ln a$

Derivative of the form $y = a^{f(x)}$

Apply natural logarithm both sides

$$\ln y = \ln a^{f(x)}$$

$$\ln y = f(x) \ln a$$

$$\frac{1}{y} \frac{dy}{dx} = f'(x) \ln a \quad \text{thus} \quad \frac{dy}{dx} = y f'(x) \ln a$$

$$\frac{d}{dx}(a^{f(x)}) = f(x) \cdot f'(x) \ln a$$

Example 36

Differentiate $y = a^{\cos x}$

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Solution

$$\text{Let } u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$$

$$y = a^u \Rightarrow \frac{dy}{du} = a^u \ln a$$

$$\frac{dy}{dx} = (a^u \ln a)(-\sin x)$$

$$\frac{dy}{dx} = -(\sin x)a^{\cos x} \ln a$$

Example 37

$$\text{Find } \frac{dy}{dx} \text{ if } y = 3^{-x^2+4x}$$

Solution

$$\text{Let } u = -x^2 + 4x \Rightarrow \frac{du}{dx} = -2x + 4$$

$$y = 3^u \Rightarrow \frac{dy}{du} = 3^u \ln 3 \text{ thus } \frac{dy}{dx} = (-2x + 4)(3^u \ln 3)$$

$$\frac{dy}{dx} = (-2x + 4)(3^{-x^2+4x}) \ln 3$$

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EXERCISE 9

1. Prove that if $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$
2. Differentiate the following with respect to x

$$(a) \ y = \sin(e^{2x})$$

$$(e) \ y = \frac{e^{x^2}}{2x^2 + \ln x}$$

$$(b) y = \tan(\ln \tan x)$$

$$(f) y = e^{\cos^{-1} x^2}$$

$$(c) y = \ln \sqrt{\frac{1 + \cos x}{1 - \cos x}}$$

$$(g) y = \sqrt[3]{e^{\sqrt{\tan x}}}$$

$$(d) y = \frac{1}{\ln x^2}$$

$$(h) \tan y = \left(\frac{3^{x+1}}{1 - 9^x} \right)$$

3. Differentiate

$$(a) y = x^{x^x}$$

$$(b) y = (\cos x)^{\sin(x^2)}$$

$$(c) y = (\sin x)^{\ln(ax+b)}$$

$$(d) x^y = e^{x-y}$$

4. Prove that $\frac{dy}{dx} = \frac{y}{x}$ if $x^m y^n = (x+y)^{m+n}$

5. Differentiate with respect to x , $y = \sin(3x)\sin(4x)$

6. Differentiate with respect to x , $y = e^{\sin 2x + \tan x}$

7. If $y = \sin^{-1}(2x^2 - 1)$, show that $\frac{dy}{dx} = \frac{2x}{\sqrt{1 - (2x^2 - 1)^2}}$

EXERCISE 10

Differentiate the following

1. $f(x) = 3^{x^2-2}$

2. $f(x) = xe^{x-4}$

3. $f(x) = 10^{x^3+x}$

4. $f(x) = 2^{\cos(x)}$

5. $f(x) = 3^{\cot x^2}$

6. $f(x) = 4^{\ln(x^2+3x)}$

7. $f(x) = \pi^{(2x+3)}$

8. $f(x) = e^{2\cos(x)\tan(x)}$

9. $f(x) = 5^{\ln(\log x^2)}$

10. $f(x) = a^{g(x)+f(x)}$

9.5. Second and higher derivatives

The term $\frac{dy}{dx}$ in differentiation is called the first derivative and sometimes denoted by y' or $f'(x)$. The second, third, fourth and n^{th}

are denoted by $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ and $\frac{d^ny}{dx^n}$ respectively. The second derivative is obtained by differentiating the first derivative, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$, the third derivative is obtained by differentiating the second derivative, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ and so forth.

Find (i) $\frac{dy}{dx}$ (ii) $\frac{d^2y}{dx^2}$ if $y = e^{x^2}$

Solution

Given $y = e^{x^2}$

$$\frac{dy}{dx} = 2xe^{x^2} \text{ and } \frac{d^2y}{dx^2} = (4x^2 + 2)e^{x^2}$$

Example 39

If $y = \sin 2x$ find $y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} + \dots$ use the series obtained find 6th terms of the series.

Solution

Given $y = \sin 2x$

$$y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} + \frac{d^4y}{dx^4} + \frac{d^5y}{dx^5} + \dots$$

$$\frac{dy}{dx} = 2 \cos 2x, \frac{d^2y}{dx^2} = -4 \sin 2x, \frac{d^3y}{dx^3} = -8 \cos 2x$$

$$y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} + \dots$$

$$= \sin 2x + 2 \cos 2x - 4 \sin 2x - 8 \cos 2x + 16 \sin 2x + 32 \cos 2x + \dots$$

From the series above

$(\sin 2x + 2 \cos 2x) - 4(\sin 2x + 2 \cos 2x) + 16(\sin 2x + 2 \cos 2x) + \dots$ This is a geometric series,

Common ratio, $r = -4 \left(\frac{\sin 2x + 2 \cos 2x}{\sin 2x + 2 \cos 2x} \right) = -4$

The sixth term, $G_6 = G_1 r^{n-1}$

$$G_6 = 1024 (\sin 2x + 2 \cos 2x)$$

Example 40

Show that $2x + (1+x^2) \frac{d^2y}{dx^2} = 0$ if $\tan y = x$

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Solution

Given $\tan y = x$

$$\sec^2 y \frac{dy}{dx} = 1$$

By product rule

$$2 \sec^2 y \tan y \frac{dy}{dx} + \sec^2 y \frac{dy}{dx} = 0$$

$$2 \tan y \left(\sec^2 y \frac{dy}{dx} \right) + \sec^2 y \frac{d^2y}{dx^2} = 0$$

$$\sec y = \sqrt{1 + \tan^2 y}$$

$$2 \tan y + (1 + \tan^2 y) \frac{d^2 y}{dx^2} = 0$$

Hence shown, $2x + (1 + x^2) \frac{d^2 y}{dx^2} = 0$

Example 41

Show that $\frac{d^2 y}{dx^2} = 2(y-1)(3y-5)$ if $y-1 = \sec^2 x$

Solution

From $y-1 = \sec^2 x$

$$\frac{dy}{dx} = 2 \sec^2 x \tan x$$

$$\frac{d^2 y}{dx^2} = 2(2 \sec^2 x \tan^2 x) + 2 \sec^2 x \sec^2 x$$

$$\frac{d^2 y}{dx^2} = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$$

But $y-1 = \sec^2 x$

$$\frac{d^2 y}{dx^2} = 4(y-1)(y-2) + 2(y-1)^2$$

$$\frac{d^2 y}{dx^2} = 2(y-1)(2(y-2) + (y-1))$$

$$\frac{d^2 y}{dx^2} = 2(y-1)(3y-5)$$

Example 42

Show that if $y = \tan^{-1} 2x$ then $4x \frac{dy}{dx} + \frac{d^2 y}{dx^2} = 0$

Solution

Given $\tan y = 2x$

$$\sec^2 y \frac{dy}{dx} = 2$$

$$2 \sec^2 y \tan y \frac{dy}{dx} + \sec^2 y \frac{d^2 y}{dx^2} = 0$$

$$2 \tan y \frac{dy}{dx} + \frac{d^2 y}{dx^2} = 0$$

$$2(2x) \frac{dy}{dx} + \frac{d^2 y}{dx^2} = 0$$

$$4x \frac{dy}{dx} + \frac{d^2 y}{dx^2} = 0$$

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Example 43

Show that $\frac{dy}{dx} + \sqrt{x^2 - 3x} \frac{d^2 y}{dx^2} = 1$ if $y^2 = x^2 - 3x$

Solution

Given $y^2 = x^2 - 3x$

$$2y \frac{dy}{dx} = 2x - 3$$

$$2 \frac{dy}{dx} + 2y \frac{d^2 y}{dx^2} = 2$$

$$\frac{dy}{dx} + y \frac{d^2y}{dx^2} = 1$$

$$\frac{dy}{dx} + \sqrt{x^2 - 3x} \frac{d^2y}{dx^2} = 1 \text{ hence shown.}$$

EXERCISE 11

1. Given $y = \frac{1}{x}$ find (i) $\frac{d^2y}{dx^2}$ (ii) $\frac{d^4y}{dx^4}$
2. Prove that if $y = Ae^{2x} + Be^x$ then $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$
3. Prove that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ then $y = (Ax + B)e^{2x}$
4. If $y = e^{2x}$ show that $\frac{d^n y}{dx^n} = 2^n y$ hence find $\frac{d^{10}y}{dx^{10}}$
5. If $y = \sin 2x + e^{2x}$ show that $\frac{d^4y}{dx^4} - 16y = 0$
6. If $y = \sqrt{x+3}$ show that $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$
7. Eliminate A and B from $y = Axe^{-3x} + Be^{-3x}$
8. Deduce that formula for $\frac{d^n y}{dx^n}$ from $y = e^{x^2}$
9. Given $y = x^x$ show that $-x\left(\frac{dy}{dx}\right)^2 + xy \frac{d^2y}{dx^2} - y^2 = 0$
10. Express y in terms of x from $\frac{dy}{dx} = \frac{d^2y}{dx^2}$

9.6. Differentiation implicit functions

So far, we have learned the differentiation of *explicit* functions using number of differentiation rules. When a function is given in such a way that dependent variable y is not given explicit in terms of independent variables the function is called *implicit function*. Consider the functions (a) $x^2 + xy^3 + y^2 = 5$ and (ii) $x + 2y - \sin 3y = 1$ are both implicit functions. The differentiation of these functions is easy using any appropriate rule of differentiation discussed above.

Example 44

If $x^2 + x^3y - \cos 2y = 4x$ find $\frac{dy}{dx}$

Solution

$$x^2 + x^3y - \cos 2y = 4x$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(x^3y) - \frac{d}{dx}(\cos 2y) = \frac{d}{dx}(4x)$$

$$2x + 3x^2y + x^3 \frac{dy}{dx} + 2 \sin 2y \frac{dy}{dx} = 4$$

$$(x^3 + 2 \sin 2y) \frac{dy}{dx} = 4 - 2x - 3x^2y$$

$$\frac{dy}{dx} = \frac{4 - 2x - 3x^2y}{x^3 + 2 \sin 2y}$$

Example 45

Find $\frac{dy}{dx}$ from $x^3 + y^3 = 4xy$

Solution

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Given $x^3 + y^3 = 4xy$

$$3x^2 + 3y^2 \frac{dy}{dx} = 4y + 4x \frac{dy}{dx}$$

$$(3y^2 - 4x) \frac{dy}{dx} = 4y - 3x^2$$

$$\frac{dy}{dx} = \frac{4y - 3x^2}{3y^2 - 4x}$$

Example 46

Find $\frac{dy}{dx}$ from $\cos(x^2y) + 3\sin(x^2y) = 3$

Solution

$$\cos(x^2y) + 3\sin(x^2y) = 3$$

$$-\sin(x^2y) \left(2xy + x^2 \frac{dy}{dx} \right) + 3\cos(x^2y) \left(2xy + x^2 \frac{dy}{dx} \right) = 0$$

$$\left(2xy + x^2 \frac{dy}{dx} \right) (3\cos(x^2y) - \sin(x^2y)) = 0$$

$$\left(2xy + x^2 \frac{dy}{dx} \right) = \frac{3}{3\cos(x^2y) - \sin(x^2y)}$$

$$x^2 \frac{dy}{dx} = \frac{3}{3\cos(x^2y) - \sin(x^2y)} - 2xy$$

$$\frac{dy}{dx} = \frac{3}{3\cos(x^2y) - \sin(x^2y)} - \frac{2y}{x}$$

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Example 47

Show that if $x\sqrt{1+y} - y\sqrt{1+x} = 0$ then $\frac{dy}{dx} = -\frac{1+y}{1+x}$

Solution

From $x\sqrt{1+y} - y\sqrt{1+x} = 0$

$$x^2(1+y) = y^2(1+x)$$

$$x^2 - y^2 = xy^2 - x^2y$$

$$(x-y)(x+y) = xy(y-x) \text{ thus } x+y = -xy$$

$$1 + \frac{dy}{dx} = -x \frac{dy}{dx} - y$$

$$(1+x) \frac{dy}{dx} = -1-y$$

$$\frac{dy}{dx} = -\frac{1+y}{1+x} \text{ hence shown}$$

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EXERCISE 12

Differentiate the following

1. $xy^2 - 2x^2y = \sin y$

2. $x^y + y^x + y^2 = 0$

3. $\sqrt{x}y + y^3 - 3x + \tan x = y^2$

4. $\ln(xy) = x^3y^2 + \sqrt{y}$

5. $x^3 - \frac{y}{x} + \cos(xy) - \sin x = 0$

6. $\log(y) + \ln xy = x^3$

7. $\sqrt{x+y^2} - x^3 = e^{xy}$

8. $(x+y^2)^3 + xy^2 = \cos(x+y^2)$

9. $2^x + xy^3 - e^y = ky^2$

10. $\cosh(x^3 + xy^2) + \sinh(2x^2)$

where k is a constant

9.7. Parametric differentiation

Let consider the functions $x(t) = at^2 + bt$ and $y(t) = 2at + b$ these two equations are equations of t , these equations are called *parametric equation*, t is called a *parameter*. In this case the derivative is given in terms of the parameters.

Example 48

Find $\frac{dy}{dx}$ if $x = 2t^3 + t^2$ and $y = t^2 + 3t$

Solution

Given $x = 2t^3 + t^2 \Rightarrow \frac{dx}{dt} = 6t^2 + 2t$

$$y = t^2 + 3t \Rightarrow \frac{dy}{dt} = 2t + 3$$

By chain rule $\frac{dy}{dx} = \left(\frac{dy}{dt} \right) \left(\frac{dt}{dx} \right)$

$$\frac{dy}{dx} = (2t + 3) \left(\frac{1}{6t^2 + 2t} \right) = \frac{2t + 3}{6t^2 + 2t}$$

Example 49

Given $x = \cos(t^2 - 1)$ and $y = \sin(t^3 + 2)$ find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dt} = 3t^2 \cos(t^3 + 2)$$

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$$\frac{dx}{dt} = -2t \sin(t^2 - 1)$$

$$\frac{dy}{dx} = \frac{3t^2 \cos(t^3 + 2)}{-2t \sin(t^2 - 1)}$$

$$\frac{dy}{dx} = -\frac{3t \cos(t^3 + 2)}{2 \sin(t^2 - 1)}$$

Example 50

Find $\frac{dy}{dx}$ if $x = \frac{2t}{t+2}$ and $y = \frac{3t}{t+2}$

Solution

$$\frac{dx}{dt} = \frac{(t+2)(2) - (2t)(1)}{(t+2)^2} = \frac{4}{(t+2)^2}$$

$$\frac{dy}{dt} = \frac{3(t+2) - 2t}{(t+2)^2} = \frac{t+6}{(t+2)^2}$$

$$\frac{dy}{dx} = \frac{t+6}{(t+2)^2} \times \frac{(t+2)^2}{4} = \frac{t+6}{4}$$

$$\frac{dy}{dx} = \frac{t+6}{4}$$

If $\frac{dy}{dx}$ exist, the second derivative is obtained by $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \times \frac{dt}{dx}$

Example 51

If $x = at^2 - b$ and $y = 2at^3 + b$ find (a) $\frac{dy}{dt}$ (b) $\frac{d^2y}{dx^2}$ in terms of t .

Solution

$$(a) \quad \frac{dy}{dt} = 6at^2 \quad \text{and} \quad \frac{dx}{dt} = 2at$$

$$\frac{dy}{dx} = \frac{6at^2}{2at} = 3t$$

$$(b) \quad \frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \times \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = 3 \times (2at) = 6at$$

EXERCISE 13

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from

1. $x(t) = 2t^3 - \frac{1}{t}$ and $y(t) = x^{-2}$

2. $y = \cos(t)$ and $x = \sin(t^2)$

3. $y = 2t^2 + \sqrt{t}$ and $x = \frac{2}{\sqrt{t}} - 3t^3$

4. $y = \tan(t)$ and $x = \sec(t)$

5. $y = \sqrt{2t^2 - t}$ and $x = (2t^2 - t)^2$

6. $y = \cos(3t - 2)$ and $x = \sin(2t^2 - 3)$

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9.8. Derivative of a function with respect to another function

We have discussed the derivative of a function with respect to a variable (i.e. x, y, z etc.). In some cases, we may be required to find the derivative of a function with respect to another function.

Suppose we have functions $f(x)$ and $g(x)$, let $u = f(x)$ and

$v = g(x)$, find $\frac{du}{dx}$ and $\frac{dv}{dx}$, therefore chain rule helps to

$$\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$$

Example 52

Differentiate $y = \sin 3x$ with respect to $v = e^{2x}$

Solution

Given $y = \sin 3x$ then $\frac{dy}{dx} = 3 \cos 3x$ and $v = e^{2x}$ then $\frac{dv}{dx} = 2e^{2x}$

Chain rule, $\frac{dy}{dv} = \frac{dy}{dx} \times \frac{dx}{dv}$

$$\frac{dy}{dv} = (3 \cos 3x) \left(\frac{1}{2e^{2x}} \right)$$

$$\frac{dy}{dv} = \frac{3 \cos 3x}{2e^{2x}}$$

Example 53

Differentiate $y = \tan^{-1} x^2$ with respect to $g(x) = \sin^{-1} x$

Solution

Given $y = \tan^{-1} x^2$ and $g(x) = \sin^{-1} x$

$$\tan y = x^2 \Rightarrow \sec^2 y \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{\sec^2 y} = \frac{2x}{1 + \tan^2 y}$$

$$\frac{dy}{dx} = \frac{2x}{1 + x^4}$$

$$\sin g = x \Rightarrow \cos g \frac{dg}{dx} = 1$$

$$\frac{dg}{dx} = \frac{1}{\cos g} = \frac{1}{\sqrt{1 - \sin^2 g}}$$

$$\frac{dg}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Chain rule $\frac{dy}{dg} = \frac{dy}{dx} \times \frac{dx}{dg}$

$$\frac{dy}{dx} = \frac{2x}{1 + x^4} \times \sqrt{1 - x^2} \quad \text{thus} \quad \frac{dy}{dg} = \frac{2x\sqrt{1 - x^2}}{1 - x^4}$$

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Example 54

Differentiate $y = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$ with respect to $u = e^{x^2 + 1}$

Solution

Given $y = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$

Let $x = \tan \theta$ therefore, $\frac{dx}{d\theta} = \sec^2 \theta$

$$\sin y = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$\sin y = \cos 2\theta$$

$$\sin y = \sin\left(\frac{\pi}{2} - 2\theta\right)$$

$$y = \frac{\pi}{2} - 2\theta \quad \text{thus} \quad \frac{dy}{d\theta} = -2$$

By chain rule $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$ thus $\frac{dy}{dx} = -2 \sec^2 \theta$

$$\frac{dy}{dx} = -2 \sec^2(\tan^{-1} x)$$

$$\frac{dy}{dx} = -2(1 + \tan^2(\tan^{-1} x)) \quad \text{thus} \quad \frac{dy}{dx} = -2 - 2x^2$$

From $u = e^{x^2+1} \Rightarrow \frac{du}{dx} = 2xe^{x^2+1}$ thus

Chain rule, $\frac{dy}{du} = -\frac{2 + 2x^2}{2xe^{x^2+1}}$

EXERCISE 14

1. Differentiate $y = \sin^2(3x)$ with respect to $x^2 + 3x$
2. Differentiate $y = \frac{x+1}{x-3}$ with respect to $x^2 + 1$
3. Differentiate $y = \tan(\cos 2x)$ with respect to $\sin(x^2)$
4. Differentiate $y = (x^3 + 2)^2(2x^3 - 2)^5$ with respect to $2x^2$
5. Differentiate $y = \cos(2x)$ with respect to $\sin 2x$
6. Differentiate $y = (x^2 + 3)\sqrt{x+2}$ with respect to x^9

7. Differentiate $y = x^{x^2}$ with respect to $\ln x$
8. Differentiate $y = \cos(x^x)$ with respect to x^2
9. Differentiate $y = \sqrt[3]{x-4x^2}$ with respect to $4x^2 - x$
10. Differentiate $y = (x^2 + 2)^{-2/5} \cos(x^2)$ with respect to $\sin(x^2)$

9.9. Application of differentiation

9.9.1. Rate of change

Rate of change of a substance is either increase or decrease of a quantity per time taken. In differentiation, the rate of change is denoted by $\frac{dq}{dt}$ where q is a quantity function and t is a time taken.

Example 55

The surface area of a sphere is given as $A = 4\pi r^2$ where A denote the area and r denote the radius. Find the rate of change of the area when radius is 3cm, given that the rate of increase of radius is 2cm/s.

Solution

Given $A = 4\pi r^2$, $\frac{dr}{dt} = 2 \text{ cm/s}$ and $r = 3\text{cm}$

Required, $\frac{dA}{dt} = ?$

By chain rule, $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

$$\frac{dA}{dr} = 8\pi r \text{ and } \frac{dr}{dt} = 2 \text{ cm/s}$$

$$\frac{dA}{dt} = 8\pi r \times 2 = 16(3)\pi = 48\pi \text{ cm}^2/\text{s}$$

Therefore, $\frac{dA}{dt} = 150.8 \text{ cm}^2/\text{s}$ (to one decimal place)

Example 56

The volume of the sphere of radius r , is $V = \frac{4}{3}\pi r^3$ and the surface area A is $A = 4\pi r^2$. The volume is increasing at the steady rate of $10 \text{ cm}^3/\text{s}$. If t is time in seconds, find

- (a) $\frac{dr}{dt}$ when r is 7 cm
- (b) $\frac{dA}{dt}$ when r is 7 cm

Solution

(a) From $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

$$\frac{dA}{dt} = 8\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot 10 = \frac{10}{4\pi r^2}$$

$$\frac{dr}{dt} = \frac{5}{2(7)^2\pi} = \frac{5}{98\pi} \text{ cm/s}$$

$$\frac{dr}{dt} = 0.0162 \text{ cm/s (to 3 significant figures)}$$

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$$\begin{aligned}
 \text{(b)} \quad \frac{dr}{dt} &= \frac{dr}{dA} \times \frac{dA}{dt} \\
 \frac{dA}{dt} &= 8\pi r \cdot \left(\frac{5}{98\pi} \right) = \frac{40r}{98} \\
 \frac{dA}{dt} &= 20/7 \text{ cm}^2/\text{s}
 \end{aligned}$$

Example 57

A container in the shape of a right circular cone of the height 10cm and base radius of 1cm is catching the drips from a leaking tap at the rate of $0.1 \text{ cm}^3/\text{s}$. Find the rate at which the surface area of water is increasing when the water is half-way up the cone.

Solution

Consider a cone below $V = \frac{1}{3} \pi r^2 h$

Consider the relation $\frac{H}{h} = \frac{R}{r} \Rightarrow \frac{10}{h} = \frac{1}{r}$ then $r = \frac{1}{10} h$

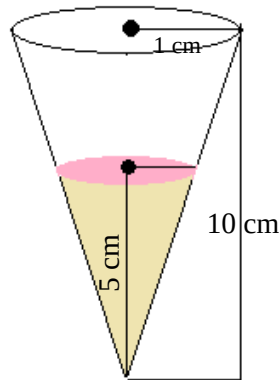
$$V = \frac{1}{3} \pi \left(\frac{1}{10} h \right)^2 h$$

$$V = \frac{1}{300} \pi h^3$$

$$\frac{dV}{dh} = \frac{\pi h^2}{100}$$

Chain rule, $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$$\text{Then } \frac{dh}{dt} = \frac{100}{\pi h^2} \times 0.1 = \frac{10}{\pi h^2}$$



Surface area of water in a cone, $A = \pi r^2$

But
$$r = \frac{h}{10}$$

Area become,
$$A = \frac{\pi h^2}{100}$$

$$\frac{dA}{dh} = \frac{\pi h}{50} \text{ thus } \frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$$

When $h = 5 \text{ cm}$

$$\frac{dA}{dt} = \frac{\pi h}{50} \times \frac{10}{\pi h^2} = \frac{1}{5h} = \frac{1}{5 \times 5}$$

$$\frac{dA}{dt} = \frac{1}{25} \text{ cm}^2/\text{s}$$

The rate at which the surface area is increasing is $0.04 \text{ cm}^2/\text{s}$

Example 58

A horse trough has a triangular cross section of height 25 cm and base 30 cm, and is 200 cm long. A horse is drinking steadily, and when the water level is 5 cm below the top it is being lowered at the rate of 1 cm/min. Find the rate of consumption.

Solution

Given $\frac{dh}{dt} = 1 \text{ cm/min}$, required $\frac{dV}{dt} = ?$

Consider the relation

$$\frac{H}{h} = \frac{B}{b} \text{ from the extract}$$

$$\frac{25}{h} = \frac{30}{b} \Rightarrow b = \frac{6}{5}h$$

$$\text{Volume, } V = \frac{1}{2}bhl$$

$$V = \frac{1}{2}\left(\frac{6}{5}h\right)hl$$

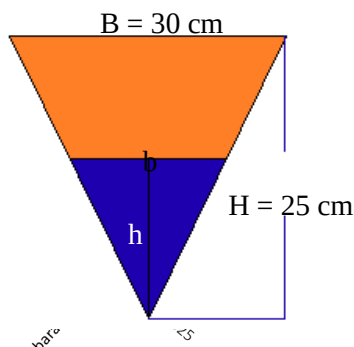
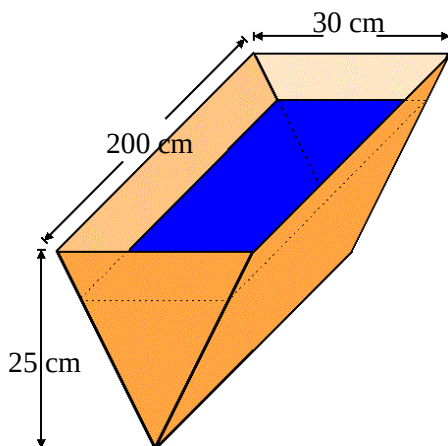
But $l = 200\text{cm}$ (given)

$$V = 120h^2$$

$$\frac{dV}{dt} = 240h \frac{dh}{dt}$$

$$\frac{dV}{dt} = 240(20)(1) = 4800 \text{ cm}^3/\text{min}$$

The rate of consumption is $4800 \text{ cm}^3/\text{min}$



9.9.2. Small change

Example 59

Find an approximate of $\sqrt{9.1}$ correct to 3 decimal places, using differentiation

Solution

Let $y = \sqrt{x}$

$$\text{Then } \delta y + y = \frac{1}{2\sqrt{x}} \delta x + \sqrt{x}$$

$$\delta y + y = \left(\frac{1}{2\sqrt{9}} \times 0.1 \right) + \sqrt{9}$$

The approximate root is $\sqrt{9.1} \approx 3.017$

Example 60

A 5% error is made in measuring the radius of the sphere. Find the percentage error in

- (a) Surface area
- (b) Volume

Solution

- (a) The surface area of a sphere, $S = 4\pi r^2$

$$\delta S = 8\pi r \delta r$$

$$\text{But } \delta r = 5r\%$$

$$\frac{\delta S}{S} \times 100\% = \frac{8\pi r}{4\pi r^2} \times \left(\frac{5r}{100} \right) \times 100\%$$

$$\frac{\delta S}{S} \times 100\% = \frac{40\pi r^2}{4\pi r^2} \% = 10\%$$

The percentage change in surface area is 10%

(b) $V = \frac{4}{3}\pi r^3$

$$\delta V = 4\pi r^2 \delta r$$

$$\frac{\delta V}{V} \times 100\% = \frac{4\pi r^2 \delta r}{\frac{4}{3}\pi r^3} \quad \text{But } \delta r = \frac{5r}{100}$$

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$$\frac{dV}{V} \times 100\% = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} \left(\frac{5r}{100} \right) \times 100\% = 15\%$$

Percentage change in volume is 15%

EXERCISE 15

- (a) A spherical balloon is blown up so that its volume increases at a constant rate of 2 cm cubic per second. Find the rate of increase of the radius when the volume of the balloon is 50 cm cubic.

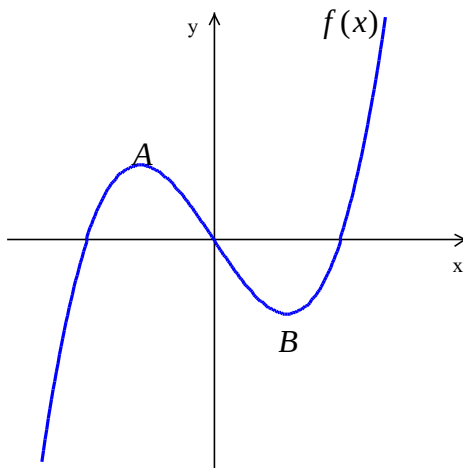
(b) A group of bacteria was placed on a piece of land and form a circular disc shape which increases (due to reproduction) in area at the rate of 5 cm square per second. Find the rate of increase of the radius when the area is 30 cm square.
- The radius (metres) of a tank increases at the rate of 0.4 every minute when it catches water. Find the increase in volume of the tank when radius equals to the height of the tank equals to 0.1 meters.
- A cubical block of ice is melting and its edge length is decreasing at the constant rate of 2 mm per second. If the block remains cubical, find the rate at which the volume is decreasing when the cube of edge length 5 cm.
- A container is in a shape of an inverted right circular cone of base radius 5 cm and height of 30 cm. If water is poured into the container at the rate of 1.5 cm^3 in every minute, find the rate at which the level of water is rising when water is 20 cm deep.
- A ladder 10 m long is leaning against a vertical wall with its lower end on horizontal ground. If the lower end is 6 m from the wall and is slipping away from the wall at a constant rate of 0.2

m per second, find the rate at which the upper end sliding down the wall.

6. A square is of edge length of 10 cm. If the sides are increasing in length at the rate of 0.5 cm per second, find the rate at which:
 - (a) The perimeter is increasing
 - (b) The area is increasing
 - (c) The length of the diagonal is increasing
7. A metal disc, 20 cm in diameter, expands on heating. If the radius is increasing at the rate of 0.01 cm per second. Find the rate at which the area of the faces is increasing.
8. Water is poured into a hemispherical bowl of radius 30 cm at the rate of $141\frac{3}{7}$ cm³ in every minute. Find how fast the water level is rising when the water is 15 cm deep at the centre. (Use the volume formula, $V = \frac{1}{3}\pi h^2(3r - h)$, where h is altitude and r is the radius)
9. An inverted right-circular cone is catching water from a tap at the rate 10 cm³ per minute. If the base radius is equal to the altitude of the cone, find the rate at which the water level is rising when the water is 8 cm deep
10. A water tank in the shape of a right circular cylinder of radius 20 m is partly filled with water. A sphere of radius 2 m is lowered into the water at the constant rate of 0.3 m per second. Find the rate at which the water level is rising when the sphere is half submerged.
11. A rectangular trough is 2 meter wide and 12 meter long. If the water is pouring in at the rate of 4 m³ in every second, find the rate at which the water level is rising.

9.9.3. Graph sketching

Using this concept of a derivative, we can easily sketch the graph of polynomials, by finding their maximum and or the minimum point or an inflexion point. The point where the graph changes from an increasing function to a decreasing function. These points are called **stationary points**.



Are the points where $\frac{d}{dx} f(x) = 0$

In the graph below, point A is the maximum turning point and point B is the minimum turning point. At these points, the graph attain the maximum at point A and minimum at point B.

9.9.4. Maximum and minimum condition

Let $f(x)$ be a function and attain its maximum at the point where $x = a$. Before point $x = a$ the function is an increasing function, thus the slope is positive, $\frac{dy}{dx} = +ve$, but after the maximum point the function become a decreasing function, thus the slope of the function is negative, $\frac{dy}{dx} = -ve$, so at the maximum point the derivative (slope), $\frac{dy}{dx}$ changes its sign from positive to negative

sign. At the point $x = a$ the slope is neither positive nor negative, therefore, $\frac{dy}{dx} = 0$

There are three conditions for maxima

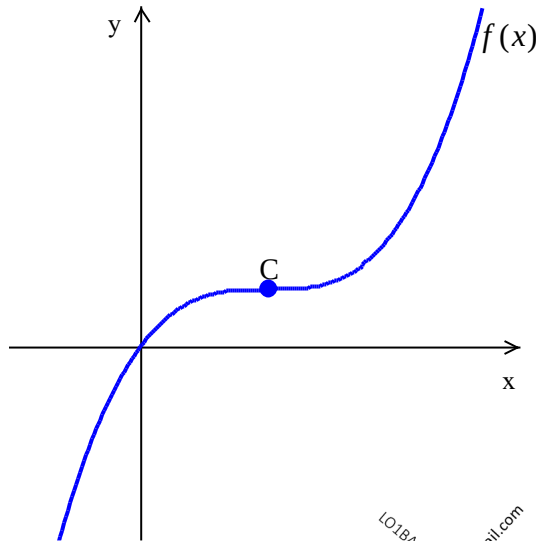
- (a) The derivative changes signs from positive to negative
- (b) The derivative at a maxima is zero, $\frac{dy}{dx} = 0$
- (c) The second derivative is negative, $\frac{d^2y}{dx^2} < 0$

Likewise, at a minimum point, the derivative changes the sign from negative to positive sign. There are three conditions for minima

- (a) The derivative changes signs from negative to positive
- (b) The derivative at a minima is zero, $\frac{dy}{dx} = 0$
- (c) The second derivative is positive, $\frac{d^2y}{dx^2} > 0$

9.9.5. Point of inflexion

Consider the graph given in figure below



Point C is neither maximum nor minimum, the slope does not change the sign as going through the point C , before point C , the slope is positive and after point C , the slope is still positive, this point is called *inflexion point*.

Conditions for point of Inflexion

- (a) The derivative does not change the sign
- (b) The derivative at the point is zero, $\frac{dy}{dx} = 0$
- (c) The second derivative is zero, $\frac{d^2y}{dx^2} = 0$

Collectively, the maxima, minima and point of inflexion are called stationary points because it the point where the slope is zero (The change point).

Determine the stationary points using second derivative test

- (a) Find the first derivative of the function, equate to zero and solve for x .
- (b) Find the second derivative of the function and substitute the value(s) of x obtained in (a) above, then if
 - (i) $\frac{d^2y}{dx^2} < 0$ (Negative) the function has the maximum value at this point.
 - (ii) $\frac{d^2y}{dx^2} > 0$ (Positive) the function has the minimum point at this point.
 - (iii) $\frac{d^2y}{dx^2} = 0$ (Zero) the function has a point of inflexion at this point.

Procedures to determine the stationary points of the curve

- (a) Find $\frac{dy}{dx}$ equate to zero and solve the equation
- (b) Test the point using the preceding and next values
- (c) Or use the second derivative test

Example 61

Determine the stationary points of the curve $y = x^3 + 3x^2 - 24x - 80$ and hence sketch the curve.

Solution

From $y = x^3 + 3x^2 - 24x - 80$

First derivative $\frac{dy}{dx} = 3x^2 + 6x - 24$

Take $\frac{dy}{dx} = 0$ and solve for x ;

$$3x^2 + 6x - 24 = 0$$

Then $x = -4$ and $x = 2$

Using first derivative test,

$$\frac{dy}{dx} = 3x^2 + 6x - 24$$

Test; $x = -4$

$$\frac{dy}{dx} = 3x^2 + 6x - 24 \text{ using } x = [-5, -3]$$

$$\frac{dy}{dx} = 3(-5)^2 + 6(-5) - 24$$

Therefore, $\frac{dy}{dx} = 21$ (+ve)

$$\frac{dy}{dx} = 3x^2 + 6x - 24$$

$$\frac{dy}{dx} = 3(-3)^2 + 6(-3) - 24$$

Therefore, $\frac{dy}{dx} = -15$ (-ve)

-5	-4	-3
21	0	-15
positive	zero	negative

The slope of the curve change sign from positive to negative, before and after $x = -4$, this means the curve has a maximum value at $x = -4$

The maximum value, $y = x^3 + 3x^2 - 24x - 80$

$$y = (-4)^3 + 3(-4)^2 - 24(-4) - 80$$

The maximum value is $y = 0$

Therefore, the turning point is $(-4, 0)$

Test; $x = 2$, using $x = [1, 3]$

Using $x = 1$

$$\frac{dy}{dx} = 3x^2 + 6x - 24$$

$$\frac{dy}{dx} = 3(1) + 6(1) - 24$$

$$\frac{dy}{dx} = -15 \quad (-ve)$$

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Using $x = 3$

$$\frac{dy}{dx} = 3x^2 + 6x - 24$$

$$\frac{dy}{dx} = 3(3)^2 + 6(3) - 24$$

$$\frac{dy}{dx} = 21 \quad (+ve)$$

The first derivative changes sign from negative to positive

1	2	3
-15	0	21
negative	zero	positive

The test show that, the curve has a minimum value at $x = 2$

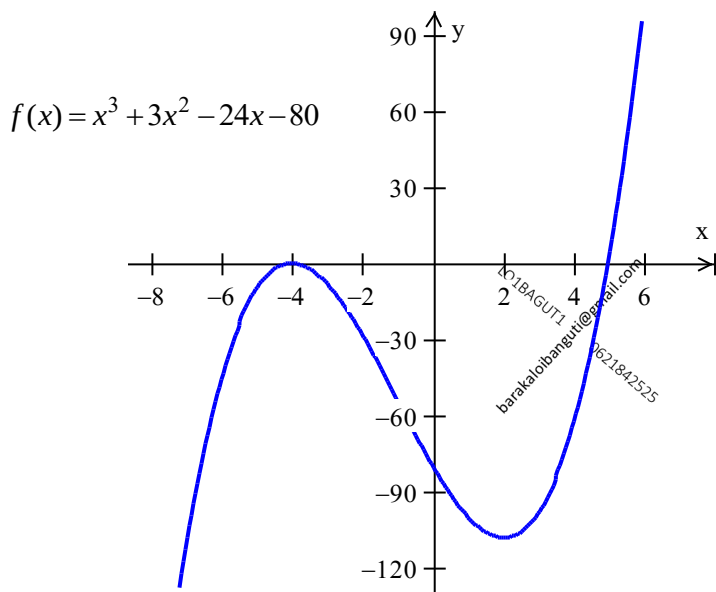
The minimum value, $y = x^3 + 3x^2 - 24x - 80$

$$y = (2)^3 + 3(2)^2 - 24(2) - 80$$

The minimum value is $y = -108$

The turning point is $(2, -108)$

To sketch the graph, we need to know, x - and y -intercepts



For x -intercept, $(x, 0)$

$$x^3 + 3x^2 - 24x - 80 = 0$$

$$x = 5 \text{ and } x = -4 \text{ (-4 root is repeated)}$$

For y -intercept, $(0, y)$

$$y = -80$$

The turning points are $(-4, 0)$ and $(2, -108)$

Example 62

Describe the turning points of the curve $y = 9x^3 - 45x^2 + 48x + 11$ and sketch the graph of y .

Solution

Using second derivative test;

From $y = 9x^3 - 45x^2 + 48x + 11$

$$\frac{dy}{dx} = 27x^2 - 90x + 48$$

Stationary point $\frac{dy}{dx} = 0$

$$27x^2 - 90x + 48 = 0$$

Therefore, $x = \frac{8}{3}$ and $x = \frac{2}{3}$

$$\frac{d^2y}{dx^2} = 54x - 90$$

When $x = \frac{8}{3}$

$$\frac{d^2y}{dx^2} = 54 > 0 \text{ this tells that, the curve attain minimum value at}$$

$$x = \frac{8}{3}$$

Substituting, $x = \frac{8}{3}$ to get $y = 9\left(\frac{8}{3}\right)^3 - 45\left(\frac{8}{3}\right)^2 + 48\left(\frac{8}{3}\right) + 11$

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The minimum value is $y = -\frac{31}{3}$

Turning point is $\left(\frac{8}{3}, -\frac{31}{3}\right)$

When $x = \frac{2}{3}$

$\frac{d^2y}{dx^2} = -54 < 0$ this tells that the curve attains maximum value at

$$x = \frac{2}{3}$$

Substituting $x = \frac{2}{3}$ to get $y = 9\left(\frac{2}{3}\right)^3 - 45\left(\frac{2}{3}\right)^2 + 48\left(\frac{2}{3}\right) + 11$

The maximum value is $y = \frac{77}{3}$

The turning point is $\left(\frac{2}{3}, \frac{77}{3}\right)$

For x- and y- intercepts

$$y = 9x^3 - 45x^2 + 48x + 11$$

For x-intercept $(x, 0)$

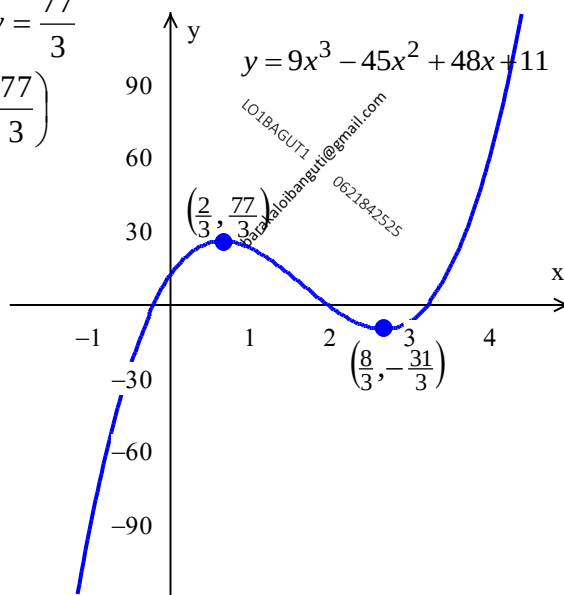
$$9x^3 - 45x^2 + 48x + 11 = 0$$

$$x = 3.23, x = -0.19$$

and $x = -1.96$

For y-intercept,

$$(0, y), y = 11$$



Example 63

Describe the stationary point of the curve $y = x^4 - 4x^3 + 3$ and sketch the graph.

Solution

Using first derivative test

Given $y = x^4 - 4x^3 + 3$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

$$\frac{dy}{dx} = 0$$

$$4x^3 - 12x^2 = 0 \text{ thus } x = 0 \text{ or } x = 3$$

At $x = 0$ testing values $x = [-1, 1]$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

-1	0	1
-16	0	-8
negative	zero	negative

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The derivative at $x = 0$ does not change the sign, this is a point of inflexion

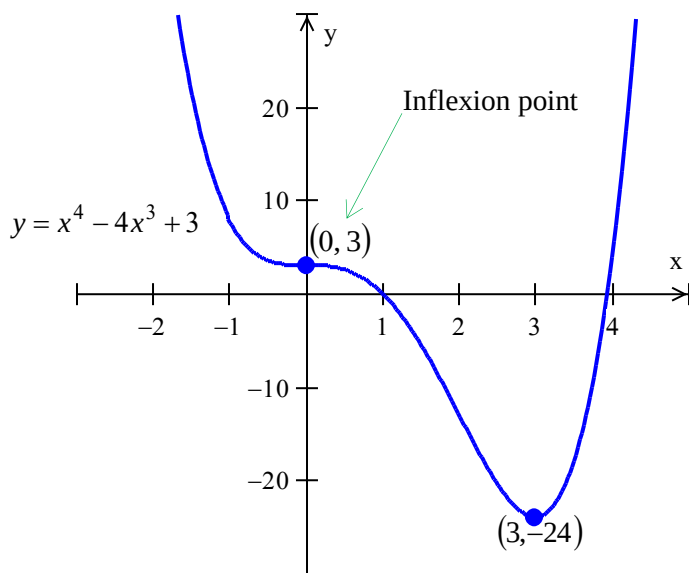
The inflexion point is, $(x, y) = (0, 3)$

At $x = 3$ testing values $x = [2, 4]$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

2	3	4
-16	0	64
negative	zero	positive

At this point, the derivative changes sign from negative to positive, implies that it has a minimum value at $x = 3$



The minimum value is, $y = -24$, the minimum turning point

9.10. Real life problems involving maxima and minima values

In economics, firms aim is to maximize profit and minimize cost of production, to attain these goals firms need to have rule that guides them, which shows the maximum possible level of output produced to maximize profit and minimize the cost.

Profit and Cost functions

Profit is the difference between the total revenue and total cost. This difference may be positive, zero or negative (Loss)

$$\text{Profit } (\pi) = \text{Total revenue} - \text{Total cost}$$

Total revenue is a gross sell of the product,

Mathematically, if q is quantity of item sold, and x is the price per item, total revenue is $TR = qx$

Example 64

Given the profit function, $\pi = -150 + 260q - 5q^2$ Tshs. determine the maximum profit

Solution

Given profit function $\pi = -150 + 260q - 5q^2$

Maximum profit $\frac{d^2\pi}{dq^2} < 0$

$$\frac{d\pi}{dq} = 260 - 10q$$

$$\frac{d\pi}{dq} = 0$$

$$260 = 10q \Rightarrow q = 26$$

Second derivative, $\frac{d^2\pi}{dq^2} = -10 < 0$ therefore, profit attain its

maximum at $q = 26$

The maximum profit is $\pi = -150 + 260(26) - 5(26)^2 \Rightarrow \pi = 3230$ Tanzania shillings.

In economics, marginal revenue (MR) is the increase in revenue that results from the sale of one additional unit of output. Marginal cost (MC) is the change in the total cost that results from producing one additional item. Marginal profit (MP) is the difference between marginal revenue and marginal cost.

Note that, $MR = \frac{d(TR)}{dx}$ and $MC = \frac{d(TC)}{dx}$

Example 65

The demand function of A to Z at a certain period of time was given by the function $x = 100 - 0.5p^2 + 0.2p^3$, find

- (a) Revenue function
- (b) Marginal revenue function

Solution

- (a) Given a demand function $x = 100 - 0.5p^2 + 0.2p^3$

Revenue = Price $\times$ Quantity

$$TR = p(100 - 0.5p^2 + 0.2p^3)$$

$$\text{Revenue function, } TR = 100p - 0.5p^3 + 0.2p^4$$

- (b) Marginal revenue, $MR = \frac{d(TR)}{dp} = 100 - 1.5p^2 + 0.8p^3$

Example 66

The revenue function of a certain firm is $R(x) = 3x - 0.002x^3$. Find the value of x that will results in maximum revenue.

Solution

Given $R(x) = 3x - 0.002x^3$

$$\frac{dR}{dx} = 3 - 0.006x^2$$

$$\frac{dR}{dx} = 0 \text{ then } 0.006x^2 = 3$$

$$x^2 = 500$$

$$x = \sqrt{500}$$

To maximize the revenue, 22 items should be produced.

Example 67

A stone is thrown upward and followed the function $f(t) = 4t - 3t^2$ metre find

- (a) The time it reached the maximum height
- (b) The maximum height reached.

Solution

$$(a) \quad \frac{df}{dt} = 4 - 9t^2$$

$$\frac{df}{dt} = 0 = 4 - 9t^2$$

Using second derivatives test, then $t^2 = \frac{4}{9}$

$$t = \frac{2}{3} \text{ (Since time is always positive)}$$

$$(b) \text{ The maximum height reached, } f\left(\frac{2}{3}\right) = 4\left(\frac{2}{3}\right) - 3\left(\frac{2}{3}\right)^3 = 1.8 \text{ m}$$

9.11. Taylor's and Maclaurin's series

Based on infinity differentiable functions Scottish mathematician James Gregory and English mathematician Brook Taylor in 1715 developed a series called **Taylor's series**. Later on, another Scottish mathematician Colin Maclaurin developed another series, which closely look like the Taylor's series, which is called **Maclaurin's series** as a special case of Taylor's series.

Consider the function below

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + \dots$$

$$f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + 20c_5(x-a)^3 + \dots$$

$$f'''(x) = 6c_3 + 24c_4(x-a) + 60c_5(x-a)^2 + \dots$$

$$f^{(4)}(x) = 24c_4 + 120c_5(x-a) + \dots$$

$$f^{(5)}(x) = 120c_5 + \dots$$

Let $x = a$, $f(a) = c_0$

$$f'(a) = c_1$$

$$f''(a) = 2c_2 \Rightarrow c_2 = \frac{1}{2} f''(a)$$

$$f'''(a) = 6c_3 \Rightarrow c_3 = \frac{1}{6} f'''(a)$$

$$f^{(4)}(a) = 24c_4 \Rightarrow c_4 = \frac{1}{24} f^{(4)}(a)$$

$$f^{(5)}(a) = 120c_5 \Rightarrow c_5 = \frac{1}{120} f^{(5)}(a)$$

$$f^n(a) = n!c_n \Rightarrow c_n = \frac{1}{n!} f^n(a)$$

Substituting

in

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

The series become;

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$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^n(a)$$

Then, let $x - a = h$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^n}{n!} f^n(a)$$

This is a Taylor's series

If $a = 0 \Rightarrow h = x$ we deduce the special case of Taylor's series called Maclaurin's series.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \frac{h^4}{4!} f''''(a) + \dots + \frac{h^n}{n!} f^n(a)$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \dots + \frac{x^n}{n!} f^n(0).$$

This series is called Maclaurin's series.

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Example 68

Use the Taylor's theorem to expand $f(x) = \cos\left(\frac{\pi}{3} + h\right)$ as far as the term containing h^5 use the first three terms of the expansion to get the value of $\cos 61.5^\circ$ correct to four decimal places, remember $1^\circ = 0.01745$ rad

Solution

$$f(x) = \cos x \Rightarrow f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f'(x) = -\sin x \Rightarrow f'\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$f''''(x) = \sin x \Rightarrow f''''\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$f''''''(x) = \cos x \Rightarrow f''''''\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{3} + h\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}h - \left(\frac{1}{2}\right)\left(\frac{1}{2!}\right)h^2 + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3!}\right)h^3 + \left(\frac{1}{2}\right)\left(\frac{1}{4!}\right)h^4 - \dots$$

$$\cos\left(\frac{\pi}{3} + h\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}h - \frac{1}{4}h^2 + \frac{\sqrt{3}}{12}h^3 + \frac{1}{48}h^4 - \frac{\sqrt{3}}{240}h^5 + \dots$$

$$\cos(60^\circ + h) = \cos(60^\circ + 1.5^\circ)$$

$$\text{Using the relation } \begin{cases} 1^\circ \Rightarrow 0.01745 \text{ rad} \\ 1.5^\circ \Rightarrow h \end{cases}$$

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Therefore, $h = 0.026175$ rad.

$$\cos(60^\circ + 1.5^\circ) = \cos 60^\circ - (0.026175) \sin 60^\circ - \frac{(0.026175)^2}{2!} \cos 60^\circ$$

$$\cos(60^\circ + 1.5^\circ) = 0.4772$$

Therefore $\cos(61.5^\circ) = 0.4772$ (correct to 4 decimal places)

Example 69

Find the Taylor's expansion of $\tan x$ about $a = \frac{\pi}{4}$ up to a term in

x^3 and hence approximate the value of $\tan 55^\circ$ correct to five decimal places.

Solution

$$f(x) = \tan x \Rightarrow \tan\left(\frac{\pi}{4}\right) = 1$$

$$f'(x) = \sec^2 x \Rightarrow \sec^2\left(\frac{\pi}{4}\right) = 2$$

$$f''(x) = 2\sec^2 x \tan x \Rightarrow 2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = 4$$

$$f'''(x) = 4\sec^2 x \tan^2 x + \sec^4 x \Rightarrow 4\sec^2\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right) + \sec^4\left(\frac{\pi}{4}\right) = 24$$

$$f(x) = f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots$$

$$f(x) = \tan\left(\frac{\pi}{4} + h\right) = \tan\left(\frac{\pi}{4}\right) + h \sec^2\left(\frac{\pi}{4}\right) + \frac{1}{2} h^2 \left(2 \sec^2\left(\frac{\pi}{4}\right) \tan\left(\frac{\pi}{4}\right)\right) + \frac{1}{6} h^3 \left(4 \sec^2\left(\frac{\pi}{4}\right) \tan^2\left(\frac{\pi}{4}\right) + \sec^4\left(\frac{\pi}{4}\right)\right) + \dots$$

$$\tan\left(\frac{\pi}{4} + h\right) = 1 + 2h + 2h^2 + 2h^3 + \dots$$

$$\tan(55^\circ) = \tan(45^\circ + 10^\circ)$$

$$\text{From } \begin{cases} 1^\circ \equiv 0.01745 \\ 10^\circ \equiv h \end{cases}$$

Therefore, $h = 0.1745$

$$\tan\left(\frac{\pi}{4} + 0.1745\right) = 1 + 2(0.1745) + 2(0.1745)^2 + 2(0.1745)^3$$

$$\tan\left(\frac{\pi}{4} + 0.1745\right) = 1.4205$$

$\tan 55^\circ = 1.4205$ (Correct to 4 decimal places)

Example 70

Use Maclaurin's series to expand $y = \ln(1+x)$ as far as the term in x^5

Solution

Given $y = \ln(1+x)$

$$f(x) = \ln(1+x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$$

$$f^{iv}(x) = -\frac{6}{(1+x)^4} \Rightarrow f^{iv}(0) = -6$$

$$f^v(x) = \frac{24}{(1+x)^5} \Rightarrow f^v(0) = 24$$

The Maclaurin's series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^v(0) + \dots$$

$$\ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24) + \dots$$

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$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

Example 71

Use Maclaurin's theorem to expand $\frac{1}{1+x}$ up to the term with x^4

Solution

$$f(x) = \frac{1}{1+x} \Rightarrow f(0) = 1$$

$$f'(x) = -\frac{1}{(1+x)^2} \Rightarrow f'(0) = -1$$

$$f''(x) = \frac{2}{(1+x)^3} \Rightarrow f''(0) = 2$$

$$f'''(x) = -\frac{6}{(1+x)^4} \Rightarrow f'''(0) = -6$$

$$f^{iv}(x) = \frac{24}{(1+x)^5} \Rightarrow f^{iv}(0) = 24$$

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Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \dots$$

Therefore, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$

Example 72

Use the series in example 71 above with the Maclaurin's series of

$f(x) = \ln(1-x)$ find the series of $\ln \sqrt{\frac{1+x}{1-x}}$

Solution

$$\text{Given } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

Maclaurin's series for $f(x) = \ln(1-x)$

$$f(x) = \ln(1-x) \Rightarrow f(0) = 0$$

$$f'(x) = -\frac{1}{1-x} \Rightarrow f'(0) = -1$$

$$f''(x) = -\frac{1}{(1-x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = -\frac{2}{(1-x)^3} \Rightarrow f'''(0) = -2$$

$$f^{iv}(x) = -\frac{6}{(1-x)^4} \Rightarrow f^{iv}(0) = -6$$

$$f^v(x) = -\frac{24}{(1-x)^4} \Rightarrow f^v(0) = -24$$

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$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^v(0) + \dots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$$

$$\text{From } \ln \sqrt{\frac{1+x}{1-x}} = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = \frac{1}{2} [\ln(1+x) - \ln(1-x)]$$

$$= \frac{1}{2} \left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \dots \right) \right]$$

$$= \frac{1}{2} \left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \dots \right]$$

Therefore, $\ln \sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \dots$

ACTIVITY

Use Maclaurin theorem or otherwise to expand $\frac{1}{(2+x)^3}$ in ascending powers of x , up to and include the term in x^3 , simplifying the coefficients

9.12. Partial derivative

Introduction of partial derivatives

Partial derivative is mostly applied in functions of several variables, the function of several variable is denoted as, $z = f(x, y)$, where both x and y are independent variables and z is a dependent variable.

To differentiate the function of several variable with respect to another variable, the other variables are treated as constants, this is what we call *partial derivative*. The partial derivative is denoted by the symbol ∂ . The partial derivative of function z with respect to x is written as $\frac{\partial z}{\partial x}$ or $f_x(x, y)$, if it is with respect to y it is written as

$\frac{\partial z}{\partial y}$ or $f_y(x, y)$ and like. In this part, we are going to deal with partial derivatives of functions of two variables, thus x and y .

Example 73

Given $z = x^2y^3 + 3xy^2 + 10x^3 - 2y$ find (a) $\frac{\partial z}{\partial x}$ (b) $\frac{\partial z}{\partial y}$

Solution

(a) $\frac{\partial z}{\partial x}$ see that, the derivative of z is with respect to x , in partial derivative this means that y is held as a constant, and the derivative of a constant is zero.

$$z = x^2y^3 + 3xy^2 + 10x^3 - 2y$$

Therefore, the derivative is $\frac{\partial z}{\partial x} = 2xy^2 + 3y^2 + 30x^2$

(b) Likewise, in the derivative $\frac{\partial z}{\partial y}$, x is held constant

The derivative become, $\frac{\partial z}{\partial y} = 3x^2y^2 + 6xy - 2$

Example 74

If $z = \cos(x^2 + 3y) + \ln(xy^3) + e^{x^2y^3}$ find

(a) $\frac{\partial z}{\partial x}$ (b) $\frac{\partial z}{\partial y}$ (c) $\frac{\partial^2 z}{\partial x \partial y}$ (d) $\frac{\partial^2 z}{\partial x^2}$

Solution

$$z = \cos(x^2 + 3y) + \ln(xy^3) + e^{x^2y^3}$$

(a) $\frac{\partial z}{\partial x} = -2x \sin(x^2 + 3y) + \frac{1}{x} + 2xe^{x^2y^3}$

(b) $\frac{\partial z}{\partial y} = -3 \sin(x^2 + 3y) + \frac{3}{y} + 3y^2e^{x^2y^3}$

$$(c) \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = -6x \cos(x^2 + 3y) + 6xy^2 e^{x^2 y^3}$$

$$(d) \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = -4x^2 \cos(x^2 + 3y) - \frac{1}{x^2} + 4x^2 e^{x^2 y^3}$$

Note that, $\frac{\partial^2 z}{\partial x \partial y}$ means differentiate with respect to y first, then

differentiate with respect to x

EXERCISE 16

1. If $z = \sin(x^2 + xy) + xy^2 - \tan x$ find (i) $\frac{\partial z}{\partial x}$ (ii) $\frac{\partial z}{\partial y}$ (iii) $\frac{\partial^2 z}{\partial x \partial y}$

2. If $z = 3x^2 y - 2xy^2 + \sqrt{xy} - \frac{x}{y}$ find (i) $\frac{\partial z}{\partial x}$ (ii) $\frac{\partial z}{\partial y}$ (iii) $\frac{\partial^2 z}{\partial x \partial y}$

3. Given $z = \sin(xy + y^3) \cos(y^2 + 2xy)$ find (i) $\frac{\partial z}{\partial x}$ (ii) $\frac{\partial z}{\partial y}$ (iii)

$$\frac{\partial^2 z}{\partial y \partial x}$$

4. If $y = \sec(x + y^2) + 3x^2 y$ find $\frac{\partial^2 z}{\partial x^2}$ (ii) $\frac{\partial z}{\partial y}$ (iii) $\frac{\partial^2 z}{\partial y^2}$

5. If $z = xy + x^2 y^2 - y^3$ find $\frac{\partial^2 z}{\partial x^2}$ (ii) $\frac{\partial z}{\partial y}$ (iii) $\frac{\partial^2 z}{\partial y^2}$

MISCELLANEOUS EXERCISE

1. The radius of a circle is increasing at the rate of 0.05 m/s, find the rate at which the area is increasing at the instant when the diameter is 10 m.
2. If $x = 2\cos 5t + 3\sin 5t$ show that $\frac{d^2x}{dt^2} = kx$ and state the value of k .
3. Find the equation of the tangent to the curve $2(x^2 + y^2)^2 = 25xy$ at the point (2, 1).
4. Find the first four terms of Maclaurin's series for $y = e^x \cos x$
5. If $f(x, y) = x^3 + x^2y^2 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$
6. Find the derivative of e^{x^2} with aspect to x using the first principle
7. A hemispherical bowl of radius 6cm contains water which is flowing into it at a constant rate when the height of water is h cm, the volume, V of water in the bowl is given by
$$V = \pi \left(6h^2 - \frac{1}{3}h^3 \right) \text{cm}^3$$
 - (a) Given that $h = 3$ cm, find the rate of change of volume of water with respect to its level.
 - (b) What is the rate of change of the volume of water if $h = 3$ cm and $t = 1$ minute.
8. If $y = (x^2 - 3x)^2$, Find $\frac{dy}{dx}$
9. Find the slope of the curve, if $x = \frac{t}{1+t}$ and $y = \frac{t^3}{1+t}$ at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$
10. If the volume of a spherical balloon increases by 2cm^3 every second, what is the rate of growth of the radius?

11. Obtain from the first principle the derivative of $y = \log_e x$
12. Find the rate at which water is being poured into a hemispherical bowl of radius 10cm if the water level is rising at the rate of 2 cm/s when the depth is 6cm. (The volume of a spherical cap of depth x cut from a sphere of radius, r is $\frac{1}{3}\pi x^2(3r - x)$)
13. Use Maclaurin's theory to expand $\ln(x+1)$ in ascending powers of x as far the term in x^4
14. Use the first principle to find the derivatives of:
- (a) $y = \frac{1}{\sqrt{x}}$ (b) $f(x) = 3x^2 + 2x + 1$
15. Differentiate the following with respect to x :
- (a) $x^3 + 3xy + y^3 = 6$ (b) $y = \frac{x^3}{\cos x}$
16. The volume of air, which is pumped into a rubber ball every second, is 4cm^3 . Given that, the volume of the ball is $V = \frac{4}{3}\pi r^3$ and its radius (r) changes with increase of air, Find the rate of change of the radius when radius is 10cm.
17. The radius of the soap burble is decreasing at the rate of 0.3cm/s. Find the rate of decrease of its surface area when the radius is 5cm.
18. A certain balloon is being blown up in such a way that its volume is increasing at a constant rate of $1.75\text{cm}^3/\text{s}$. Find the rate of increase of the radius when the volume of the balloon is 180cm^3 .
19. A particle moves following the path $S = 4 + 4t - 2t^2$, find the value of t and S when the particle is at maximum point.
20. Find the stationary points or points of inflexion of $y = \frac{x(x-1)}{x-2}$ and sketch the graph

- 21.** Find the coordinates at which point of inflexion occurs on the curve $y = e^{-2x}(2x^2 + 2x + 1)$
- 22.** Show that the equation of the tangent to the curve $x = 2a \cos^3 t$, $y = a \sin^3 t$, at any point $P(0 \leq t \leq \pi/2)$ is $x \sin t + 2y \cos t - a \sin 2t = 0$
- 23.** If $y = (\sin^{-1} x)^2$, show that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 2$.
- 24.** Form a differential equation by eliminating the constants A and B in the equation $y = Ae^{4x} + Be^{-3x}$.
- 25.** If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $(1 - x^2) \frac{dy}{dx} = xy + 1$.
- 26.** (a) If $y = \ln\left(\frac{x-1}{x+1}\right)$ find $\frac{dy}{dx}$
 (b) If $y = \frac{ax+b}{cx+d}$, show that $2\left(\frac{dy}{dx}\right)\left(\frac{d^3 y}{dx^3}\right) - 3\left(\frac{d^2 y}{dx^2}\right)^2 = 0$
- 27.** If $y = 2xe^{-3x}$ show that $\frac{d^2 y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$
- 28.** Differentiate from the first principle $f(x) = \frac{1}{x} + \cos 3x$
- 29.** Use the Taylor's theorem to obtain the series expansion for $\cos\left(\frac{\pi}{3} + h\right)$ stating the terms including that in h^3 . Hence obtain the value for $\cos 61^\circ$ giving your answer correct to six decimal places'
- 30.** Show whether the line $2x - y = 0$ and $4x^2 - 4xy + y^2 + 4x - 8y + 10 = 0$ are orthogonal.
- 31.** Given $z = x^2 + xy + y^2 - 5$ find $\frac{\partial z}{\partial y}$

32. If $y = x^{\sin x} + (\sin x)^x$ find $\frac{dy}{dx}$
33. Differentiate $v = \tan^{-1} \left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}} \right)$ with respect to $u = \cos^{-1}(x^2)$
34. If $z = \sin(3x + 2y)$ find $\frac{\partial z}{\partial y}$
35. Use the calculus technique; find an approximate value of $\sqrt{16.08}$ correct to 5 decimal places.
36. Given $y = \frac{1-t^2}{1+t^2}$ and $x = \frac{2t}{1+t^2}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
37. A gardener has 200m of fencing wire. He wishes to build a rectangular field entirely enclosed by the wire.
- What dimensions should he make the field to maximize the area?
 - What is the maximum area?
38. A container in the shape of a right circular cone of height 10 cm and base radius 1 cm is catching the drips from the tap leaking at the rate of $0.2 \text{ cm}^3/\text{s}$. Find
- The rate of height when is half-way up the cone
 - The rate at which the surface area of water is increasing when water is half way up the cone
39. A tank shaped as a right cylinder is leaking at the rate of $10 \text{ cm}^3/\text{s}$, if the radius of the tank is 30 cm and height is 120 cm, find the rate at which the height of water is decreasing when water is third quarter the height of the tank.
40. The length of the sides of the rectangular sheet of metal are 8 cm and 3 cm. A square of side x is cut from each corner of the sheet and the remaining pieces is folded to make an open box.

- (a) Show that the volume of the box formed is
 $V = (4x^3 - 22x^2 + 24x) \text{ cm}^3$
- (b) Find the value of x for which the volume of the box is maximum.
- (c) Find the maximum volume of the box
- 41.** Use the Taylor's theorem to expand $\cos\left(\frac{\pi}{3} + 2h\right)$ in increasing power of h as far as the term in h^4 . Hence estimate $\cos 48^\circ$ correct to 5 decimal places.
- 42.** Differentiate with respect to x ; $y = 3^{\cos^3 2x}$
- 43.** A tank in the form of an inverted cone having an altitude of 18 m and a base radius of 3 m long is placed closer to the water pipe. If water is flowing into the tank at the rate of 2 m^3 per minute, how fast is the water level rising when the water is 12 m deep?
- 44.** Differentiate $z = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $y = \sin^{-1}\left(\sqrt{1-x^2}\right)$
- 45.** Differentiate $y = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $x = \cos^{-1}(4t^3 - 3x)$
- 46.** Find $\frac{dy}{dx}$ if (a) $y = \cos x^\circ$ (b) $y = \tan x^\circ$ (c) $y = \sin x^\circ \sec^2 x^\circ$
- 47.** Use the Maclaurin's series expand the following
 (a) $\cos x$ (b) e^x (c) $\tan x$
- 48.** Expand $\cos\left(\frac{\pi}{6} + h\right)$ using Taylor's series up to the 5th term, and estimate the value of $\cos(34^\circ)$ correct to 6 decimal places

49. Use the Taylor's series to expand $f(x) = \sqrt[3]{x}$, up to the 4th term, use the result up to the 4th term to estimate the value of $\sqrt[3]{5.2}$ correct to 4 decimal places.
50. Use the Taylor's polynomial to expand $f(x) = \frac{1}{x^2}$ up to the 5th term and use the first 3 terms to approximate the value of $\frac{1}{12.4609}$ correct to 4 decimal places, hence calculate the absolute error.
51. Use the Taylor's polynomial to show that $\sinh x = \frac{e^x - e^{-x}}{2}$
52. Use the Maclaurin's series to expand $f(x) = e^{x^2}$ up to 5th term and evaluate $\int_0^2 e^{x^2} dx$
53. Expand $y = \tan^{-1} x^2$ using Taylor's series, up to 4th term.
54. Find the order 3 Taylor's polynomial of $f(x) = (1+x)^p$ about $a = \frac{1}{2}$ and use it to estimate the value of $(8.12)^3$
55. Obtain the first five terms of the expansion $\ln\left(\frac{1+x^2}{1-x^2}\right)$ using Taylor's polynomial when $a=0$ and use it to find the approximate value of $\ln\left(\frac{5}{4}\right)$
56. Use the Maclaurin's series to show that $\ln\left(x + \sqrt{1+x^2}\right) = x - \frac{x^3}{6}$
57. Simplify the expansion of $\sin(\cos x)$ up to the third term.
58. If $z(x+y) = x^2 + y^2$ show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

59. Given $z = x^2 + 3xy + y - 1$ find (a) $\frac{\partial z}{\partial x}$ (b) $\frac{\partial z}{\partial y}$
60. If $z = 4x^2 - 8xy^4 + 7y^4 - 5$ find
- (a) $\frac{\partial z}{\partial x}$ (b) $\frac{\partial z}{\partial y}$ (c) $\frac{\partial^2 z}{\partial x \partial y}$ (d) $\frac{\partial^3 z}{\partial x \partial^2 y}$
61. (a) Describe the stationary points of the curve $y = x^4 - 8x^2$ and hence sketch the graph
- (b) Differentiate $u = (x^2 - 3x)^{e^{2x}}$ with respect to $v = \sec^{-1}(x^2 - 1)$
62. Use the concept of differentiation to sketch the graph of
- (a) $y = x^3 + x^2 - 8x - 12$ (c) $y = e^x + e^{-2x}$
- (b) $y = \sqrt{x} - x^2$
63. Find the dimension of a rectangle with perimeter 1000 metres so that the area of the rectangle is a maximum.
64. A gardener has 8km of fencing wire, and wishes to fence a rectangular zoo. One boundary of the zoo is the bank of straight river. What are the dimensions of the zoo so that the area is maximum?
65. A square sheet of cardboard with each side k cm is to be used to make an open-top box by cutting a small square of cardboard from each of the corners and bending up the sides. What is the sides length of the small squares if the box is to have as large as volume as possible?
66. Gas Company wants to run a pipeline from a point A on the shore to a point B on an island, which is 6 kilometres from the shore. It costs Tshs. 40 million per kilometre to run the pipeline on shore, and Tshs. 50 million per kilometre to run it underwater. There is a point B' on the shore so that BB' is at right angle to AB'. The straight shoreline is the line AB' The distance AB' is 9 kilometres. Find how the pipeline should be laid to minimize the cost?

67. An upturned cone with semi-vertical angle 45° is being filled with kerosene at a constant rate of $30 \text{ cm}^3/\text{min}$. When the depth of the kerosene is 60 cm, find the rate at which
- The depth, h of kerosene is increasing
 - The radius, r of the surface of kerosene is increasing
 - The surface area, s of kerosene is increasing
68. Sketch the graph of $f(x) = 3x^4 - 16x^3 + 18x^2$ determine
- The maximum turning point(s).
 - The local and absolute minimum points
69. If $y = \frac{\sin x}{x^2}$ show that $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + (x^2 + 2)y = 0$
70. Prove that the relation $y = x^3 - 6x^2 + 18x + 5$ has maxima nor minima. Find the value of the relation when the rate of increase is minimal.
71. Differentiate $h(x) = 4\sec\sqrt[3]{x}$ with respect to $g(x) = 4\sec\sqrt{x}$
72. Find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(-1, -2)$ if $x^2 - 3xy + 2y^2 - 2x = 3$
73. If $x = \frac{2t}{t+2}$ and $y = \frac{3t}{t+3}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $\left(\frac{2}{3}, \frac{3}{4}\right)$
74. A rectangular block has a square base whose length is x centimetres. Its total surface area is 150 cm^2 .
- Show that the volume of the block is $\frac{1}{2}(75x - x^3) \text{ cm}^3$
 - Calculate the dimensions of the block when its volume is maximum.
75. It is given that the function $y = x^4 - 62x^2 + ax + 9$ attains its maximum at $x = a$ find the value of a .
76. Find the largest possible area of the right-angled triangle whose hypotenuse is 6 cm long.

77. A string of length 28 cm is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?
78. A window is in the form of a rectangle with a semi-circle on one of its end. If the perimeter of the window is 120cm, find the dimensions, which make the maximum area of the window.
79. Differentiate $y = \sqrt{x + \tan \sqrt{x + \tan \sqrt{x + \tan \sqrt{x + \tan \sqrt{x + \dots \infty}}}}$
80. If A , B and n are constants and $y = A \cos n\theta + B \sin n\theta$, show that
$$\frac{d^2 y}{dx^2} + n^2 y = 0$$
81. Total cost C , in Tshs of producing and marketing x units of a product is given by $C = \frac{x^3}{50} + 4x + 450$ find the marginal cost when 250 units are produced.
82. A 8m ladder rests against a vertical wall with its lower end on the horizontal ground. If the lower end is slipping away from the wall, find the rate of change of the distance the upper end is above ground with respect to the distance the lower end is from the wall when the lower end is 4 m from the wall.
83. The pressure and volume of a gas at a constant temperature are related by the equation $PV = C$, where C is constant (Boyle's Law). Show that the rate of change of P with respect to V is given by
$$\frac{dP}{dV} = -k \frac{1}{V^2}$$
84. Show that the rate of change of the area of a circle with respect to its circumference is equal to its radius.
85. (a) The school has an adjustable electric fence that is 100 m long. The school uses this fence to enclose a rectangular campus on three sides, the fourth side being a brick wall. Find the maximum area the school can enclose using this fence.

(b) A rectangular trough is 3m wide and 10m long. If the water is leaking out the trough at the rate of $3 \text{ cm}^3/\text{s}$, find the rate at which the water level is changing.

- 86.** The resistance R ohms of a parallel connection of two resistors of resistance R_1 and R_2 ohms respectively is given by $R = \frac{R_1 R_2}{R_1 + R_2}$.

Due to overheating, R_1 and R_2 are increasing at the rate of 0.01 and 0.02 ohms per minutes respectively. Find the rate at which R is changing when R_1 and R_2 are 30 and 50 ohms respectively.

- 87.** A cubical block of ice is melting and its edge length is decreasing at the constant rate of 4 mm per minute. If the block remains cubical, find the rate at which the volume is decreasing when the cube is of the length 20 cm.

- 88.** The two equal sides of an isosceles triangle are of length 6 cm. If the angle between them is increasing at the rate of 0.3 radians per second, find the rate at which the area of the triangle is increasing when the angle between the equal sides is $\frac{\pi}{3}$.

- 89.** (a) If $x^2 y - 4y = 1$ show that $\frac{dy}{dx} = -\frac{2x}{(x^2 - 4)^2}$

(b) If $y = \sin(m \sin^{-1} x)$ show that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0$

(c) Differentiate $f(x) = \ln \left(\sqrt{\frac{(x-2)^3}{x^2 + 5}} \right)$

- 90.** If $y = (1 - 3x)e^{mx}$ find (i) $\frac{dy}{dx}$ (ii) $\frac{d^2 y}{dx^2}$ (iii) The equation

connecting $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$

- 91.** Show that if $y = \log_e \cot \left(\frac{\pi}{4} - \frac{x}{2} \right)$ then $\frac{dy}{dx} = \sec x$

92. Use the Maclaurin's theorem find the first three terms of the expansion of $\sin x$ and $\cos x$ hence or otherwise, prove that the first two non-zero terms in the expansion of $\tan x$ in ascending powers of x are $x + x^3/3$, and deduce the approximated value of

$$\lim_{x \rightarrow 0} \left(\frac{x \sin x}{\cos x} \right)$$

93. Show that the maximum value of the function $f(x) = (a-x)(x-b)$ is $\frac{1}{4}(a-b)^2$ find the turning point of $f(x)$ when $a=4$ and $b=1$ hence sketch the graph of $f(x)$.

94. The parametric equations of the curve are $x = t + 2 \ln t$ and $y = 2t - \ln t$ where t takes all positive values

(a) Express $\frac{dy}{dx}$ in terms of t

(b) Find the equation of the tangent to the curve at the point where $t=1$

(c) The curve has one stationary point. Show that the y -coordinate of this point is $1 + \ln 2$ and determine whether this point is maximum, minimum or inflexion.

95. The curve $y = e^x + 4e^{-2x}$ has one stationary point

(a) Find the x -coordinate of this point (leave your answer in logarithm form).

(b) Determine whether the stationary point is a maximum or a minimum point

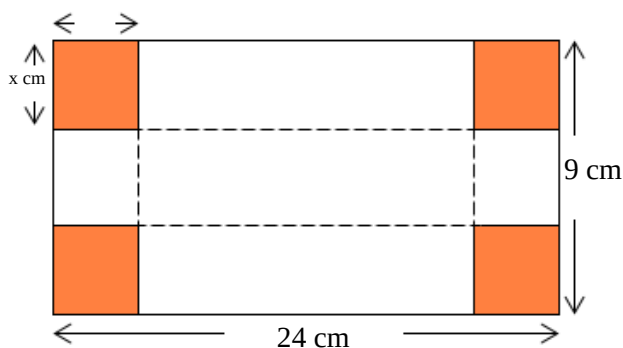
96. Differentiate with respect to x

(i) $y = \frac{\cos x - \sin x}{\cos x + \sin x}$ and simplify your answer

(ii) $y = \log_{10} x$

97. The diagram below shows a rectangular sheet of metal 24 cm by 9 cm

x cm



- (a) Find the value of x for which the volume of the box is maximum.
- (b) Find the maximum volume of the box.
- 98.** (a) A body is moving in a straight line with a retardation proportional to the square of its velocity. With the usual notation express dv/dt in terms of v .
- (b) Initially the body has a velocity of 2000 m/sec . find how far it has travelled after 3 sec., if its velocity is halved in that time. (Give the answer correct to 3 significant figures).
- 99.** If $y^2 = x^2(x-2)$, obtain an expression for dy/dx in terms of x , and hence show that on the graph of y against x there are no turning points. Show that when $x = 2\frac{2}{3}$, $dy/dx = \pm\sqrt{6}$ and $d^2y/dx^2 = 0$. Sketch the graph of y against x , paying special attention to the point $(2, 0)$ and to the points where $d^2y/dx^2 = 0$.
- 100.** Given that $y = (x^2 + 4)e^{-\frac{1}{2}x}$, show that $\frac{dy}{dx} \leq 0$ for all values of x .
Find the value of x for which
- (a) $\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} = 0$
- 101.** A printer is to use a page of 108 square inches with 1-inch margins at the sides and bottom and a $\frac{1}{2}$ -inch margin at the top.

What dimensions should the page be so that the area of the printed matter will be a maximum?

- 102.** A trough is 10 meters long and its ends have the shape of isosceles, with the base 3 meters and have a height of 1 meter. If the trough is being filled with water at a rate of $12 \text{ m}^3/\text{s}$, how fast is the water level rising when the water is 60 centimetres deep?

- 103.** (a) Differentiate $f(x) = \sin\left(x - \frac{\pi}{12}\right)$ by first principle.

(b) Given that $y = \frac{\sin^2 x \tan^4 x}{(x^4 + 1)^2}$ find $\frac{dy}{dx}$

- (c) Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $8/27$ of the volume of the sphere.
- (d) Water is dripping out from a conical funnel at a uniform rate of $4 \text{ cm}^3/\text{sec}$ through a tiny hole at the vertex in the bottom. When the slant height of the water is 3 cm , find the rate of decrease of slant height of water-cone. Given that the vertical angle of the funnel is 120° .

- 104.** Determine the turning point of the curve with equation

$$y = \frac{3e^x}{x^2}$$

- (a) Determine whether the point is maximum or minimum
- (b) If a straight line from the turning point to x – intercept 15, find the area of the shape formed with x -axis, when the other line from the origin intersects with the curve at a turning point.
- 105.** A box with a square base and open top must have a volume of $32,000 \text{ cm}^3$. How do you find the dimensions of the box that minimize the amount of material used?

ADVANCED MATHEMATICS

DIFFERENTIATION

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