

Find the volume of the solid in the first octant bounded by the planes $x^2 + y^2 = 4$ and $y + z = 3$

$$D = \{(x, y, z): 0 \leq x \leq 2; 0 \leq y \leq \sqrt{4 - x^2}; 0 \leq z \leq 3 - y\}$$

$$\Rightarrow V = \iiint dx dy dz = \int_0^2 dx \int_0^{\sqrt{4-x^2}} dy \int_0^{3-y} dz$$

OR

$$D = \{(x, y, z): 0 \leq x \leq \sqrt{4 - y^2}; 0 \leq y \leq 2; 0 \leq z \leq 3 - y\}$$

$$\Rightarrow V = \int_0^{\sqrt{4-y^2}} dx \int_0^2 dy \int_0^{3-y} dz$$

OR

$$D = \{(x, y, z) = (r \cos \theta, r \sin \theta, z) : 0 \leq r \leq 2; 0 \leq \theta \leq \frac{\pi}{2}; 0 \leq z \leq 3 - r \sin \theta\}$$

$$\Rightarrow V = \iiint r dz dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{3-r \sin \theta} r dz dr d\theta$$

SOLUTION:

First method

$$D = \{(x, y, z): 0 \leq x \leq 2; 0 \leq y \leq \sqrt{4 - x^2}; 0 \leq z \leq 3 - y\}$$

Therefore

$$V = \iiint dx dy dz = \int_0^2 dx \int_0^{\sqrt{4-x^2}} dy \int_0^{3-y} dz$$

$$\begin{aligned}
 V &= \iiint dz dy dx = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{3-y} dy dx dz = \int_0^2 \left(\int_0^{\sqrt{4-x^2}} (3-y) dy \right) dx \\
 &= \int_0^2 \left(3y - \frac{y^2}{2} \right) \Big|_0^{\sqrt{4-x^2}} dx = \int_0^2 3\sqrt{4-x^2} dx - \frac{1}{2} \int_0^2 (4-x^2) dx \\
 &\Rightarrow V = 3H_m - \frac{1}{2} \int_0^2 (4-x^2) dx
 \end{aligned}$$

where

$$H_m = \int_0^2 \sqrt{4-x^2} dx$$

In the integral H_m , let

$$x = 2\sin\varphi \Rightarrow dx = 2\cos\varphi d\varphi$$

New limits:

$$x = 0; \sin\varphi = 0 \Rightarrow \varphi = 0$$

$$x = 2; \sin\varphi = 1 \Rightarrow \varphi = \frac{\pi}{2}$$

Thus

$$\begin{aligned}
 H_m &= \int_0^2 \sqrt{4-x^2} dx = \int_0^{\pi/2} \sqrt{4-4\sin^2\varphi} (2\cos\varphi d\varphi) = \int_0^{\pi/2} 4\cos^2\varphi d\varphi \\
 &= 4 \int_0^{\pi/2} \cos^2\varphi d\varphi = 4 \int_0^{\pi/2} \left(\frac{1}{2} \right) (\cos 2\varphi + 1) d\varphi = 2 \int_0^{\pi/2} (\cos 2\varphi + 1) d\varphi \\
 &= 2 \left(\frac{1}{2} \sin 2\varphi + \varphi \right) \Big|_0^{\pi/2} = \sin 2\varphi + 2\varphi \\
 &= \sin \pi + \pi = \pi \Rightarrow H_m = \pi
 \end{aligned}$$

Finally;

$$V = 3H_m - \frac{1}{2} \int_0^2 (4 - x^2) dx = 3\pi - \frac{1}{2} \left(4x - \frac{x^3}{3} \right) \Big|_0^2 = 3\pi - \frac{1}{2} \left(8 - \frac{8}{3} \right) = 3\pi - \frac{8}{3} = 6.76$$

$$\Rightarrow V = \int_0^2 dx \int_0^{\sqrt{4-x^2}} dy \int_0^{3-y} dz = 6.76$$

ALTERNATIVE METHOD-1

$$D = \{(x, y, z): 0 \leq x \leq \sqrt{4 - y^2}; 0 \leq y \leq 2; 0 \leq z \leq 3 - y\}$$

$$V = \int_0^{\sqrt{4-y^2}} dx \int_0^2 dy \int_0^{3-y} dz$$

$$\Rightarrow V = \int_0^{\sqrt{4-y^2}} dx \int_0^2 dy \int_0^{3-y} dz$$

$$\Rightarrow V = \int_0^2 \left(\int_0^{\sqrt{4-y^2}} (3 - y) dx \right) dy = \int_0^2 \left(3\sqrt{4 - y^2} - y\sqrt{4 - y^2} \right) dy = 3\pi - H_k$$

$$\Rightarrow V = 3\pi - H_k$$

where

$$H_k = \int_0^2 y\sqrt{4 - y^2} dy$$

In this integral H_k let;

$$u = 4 - y^2 \Rightarrow du = -2y dy \Rightarrow y dy = -\frac{1}{2} du$$

New limits of integration:

$$y = 0; u = 4$$

$$y = 2; u = 0$$

$$\begin{aligned}\Rightarrow H_k &= \int_0^2 y\sqrt{4-y^2}dy = \int_4^0 u^{1/2}\left(-\frac{1}{2}du\right) = +\frac{1}{2}\int_0^4 u^{1/2}du = \left(\frac{1}{2}(u^{3/2})\left(\frac{2}{3}\right)\right)\bigg|_0^4 \\ &= \left(\frac{1}{3}u^{3/2}\right)\bigg|_0^4 = \frac{1}{3}(\sqrt{64}) = \frac{8}{3}\end{aligned}$$

$$H_k = \frac{8}{3} \Rightarrow V = 3\pi - H_k = 3\pi - \frac{8}{3} = 6.76$$

$$\Rightarrow V = \int_0^{\sqrt{4-y^2}} dx \int_0^2 dy \int_0^{3-y} dz = 6.76$$

ALTERNATIVE METHOD-2

$$D = \{(x, y, z) = (r\cos\theta, r\sin\theta, z) : 0 \leq r \leq 2; 0 \leq \theta \leq \frac{\pi}{2}; 0 \leq z \leq 3 - r\sin\theta\}$$

Using triple integration with cylindrical polar coordinates

$$x = r\cos\theta, y = r\sin\theta, z = 3 - r\sin\theta$$

$$0 \leq z \leq 3 - r\sin\theta$$

$$x^2 + y^2 = r^2; 0 \leq r \leq 2$$

$$0 \leq \theta \leq \frac{\pi}{2}$$

$$\begin{aligned}V &= \iiint r dz dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{3-r\sin\theta} r dz dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\int_0^2 r(3 - r\sin\theta) dr \right) d\theta = \int_0^{\frac{\pi}{2}} \left(\int_0^2 (3r - r^2\sin\theta) dr \right) d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\left(\frac{3r^2}{2} - \frac{r^3}{3} \sin\theta \right) \bigg|_0^2 \right) d\theta\end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\frac{\pi}{2}} \left(6 - \frac{8}{3} \sin \theta\right) d\theta = \left(6\theta + \frac{8}{3} \cos \theta\right) \Big|_0^{\frac{\pi}{2}} = 6\left(\frac{\pi}{2}\right) + \frac{8}{3} \cos\left(\frac{\pi}{2}\right) - \left(0 + \frac{8}{3} \cos 0^\circ\right) \\
 &= 3\pi + 0 - \frac{8}{3} = 3\pi - \frac{8}{3} = \mathbf{6.76}
 \end{aligned}$$

Hence;

$$\begin{aligned}
 V &= \int_0^2 dx \int_0^{\sqrt{4-x^2}} dy \int_0^{3-y} dz = \int_0^{\sqrt{4-y^2}} dx \int_0^2 dy \int_0^{3-y} dz \\
 &= \int_0^{\pi/2} d\theta \int_0^2 r dr \int_0^{3-r\sin\theta} dz = \mathbf{6.76}
 \end{aligned}$$