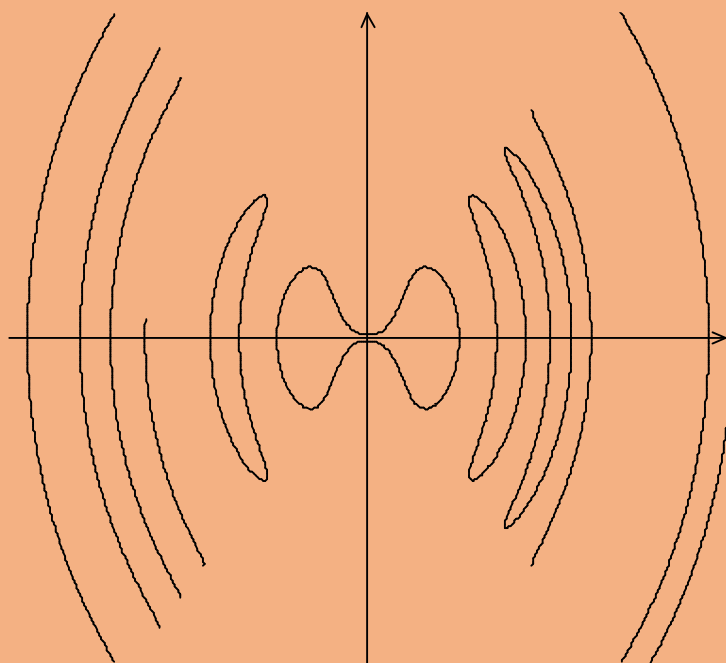


ADVANCED MATHEMATICS

FUNCTIONS



BARAKA

LOIBANGUTI

The author

Name: Baraka Loibanguti

Email: barakaloibanguti@gmail.com

Tel: +255 621 842525 or +255 719 842525

Read this!

- This book is not for sale.
- It is not permitted to reprint this book without prior written permission from the author.
- It is not permitted to post this book on a website or blog for the purpose of generating revenue or followers or for similar purposes. In doing so you will be violating the copyright of this book.
- This is the book for learners and teachers and its absolutely free.

To my Lord and Savior Jesus Christ!

FUNCTIONS

Chapter

5

Function is a rule, which assign each element of set A to a unique element of set B , where set A is referred as domain and set B is range. Such that one value of range can have more than one value of domain but not vice versa. The illustration below explains more about functions

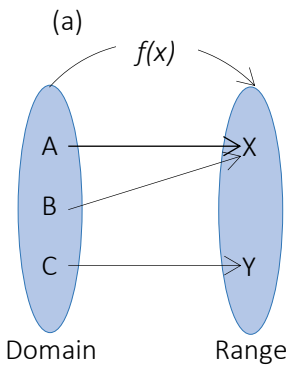


Illustration (a) above show that each member of domain set has exactly one value of range set. Therefore, this is a function.

Example: $f(x) = x^2$

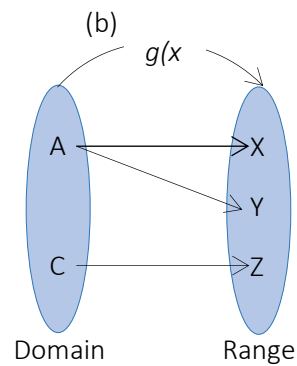


Illustration (b) above show that each member of domain set corresponds to the values of range set, but one value of domain has more than one value of range, because of this, this is not a function.

Example: $g(x) = \sqrt{x}$

A function is a relation for which there is only one value of y corresponding to any value of x .

Domain is a set of independent variables of the function while range is the set of dependent variables of the function, or is the corresponding value of the domain.

The domain of a relation is the set of all the x values for which there exists at least one y value according to that relation. The range is the set of all the y values, which can be obtained using at least one x value.

The function is denoted by $f:x \rightarrow y$ or $f(x)=y$ where x is an independent variable (domain) and y is a dependent variable (range).

A variable is a label which we allow to change and become any element of some set of numbers. For example, on a menu in a restaurant price is a variable on the set of real numbers, since for any menu item the manager can choose any price he or she feels like (with the aim of staying in business). Most often, a variable will be a letter which can take on any value in some set of numbers. In this textbook we will only use real variables, which may take on the value of any real number.

Graphs of functions

Graphs are representation of functions in drawings; in graphs, we can determine the characteristics or nature of the graph by expressing the relation between domain and range on a Cartesian plane. If the domain and range of a function are all real numbers, we could not draw the entire graph of the function, normally we only use few values of domain to get the corresponding values of range which when plotted can give a clear nature of the function.

Graphs of polynomial functions

A polynomial function is defined as $f(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_2x^2+a_1x+a_0$ where n is a non-negative integer and a is an integer and $a_n \neq 0$

For all polynomials, domain is a set of all real numbers and range is a set of all real numbers (except for constant function), if no restrictions given.

Polynomials are classified according to their degrees

- (a) Polynomial of degree zero, this is called a constant function,
 $f(x)=cx^0=c$ where c is any real number.
- (b) Polynomial of degree 1, this is called a linear function, $f(x)=ax+b$,
 where $a \neq 0$, here a is the gradient of the function and b is called y -intercept
- (c) Polynomial of degree 2, this is called a quadratic function,
 $f(x)=ax^2+bx+c$ where $a \neq 0$
- (d) Polynomial of degree 3, this is called a cubic function,
 $f(x)=ax^3+bx^2+cx+d$ where $a \neq 0$

- (e) The polynomial with degree 4 is called quartic functions, $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ where $a \neq 0$, moreover, functions with more than 4 degrees are collectively called higher degree polynomials.

Graphs of polynomials

Graphs of linear functions

Let consider the linear function of the type $f(x) = ax + b$. To plot the graph of this function, two ways may be used

- Using the table of values
- Using x and y -intercepts

One thing to note here is the graph of all linear functions are straight lines with the slope a and cut y -axis at b (refer $f(x) = ax + b$). The table of values is constructed using assumed value of x , we can assume more than one value or many values of x as possible, since the curve is a straight line we can just assume few values of x and compute the corresponding values of y and joint the points with a ruler extending beyond the points.

Example 1

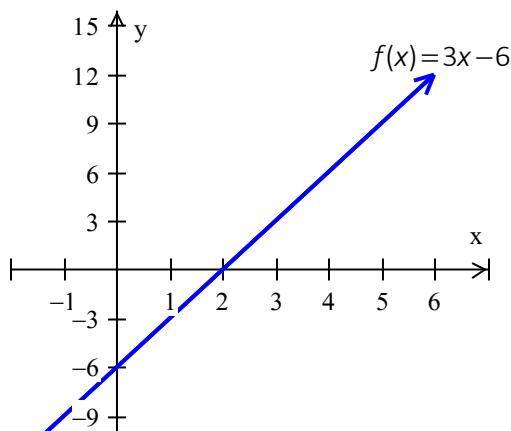
Draw the graph of $f(x) = 3x - 6$

Solution

Table of values using assumed domain values

x	-3	3
y	-15	3

Graph of $f(x) = 3x - 6$



Example 2

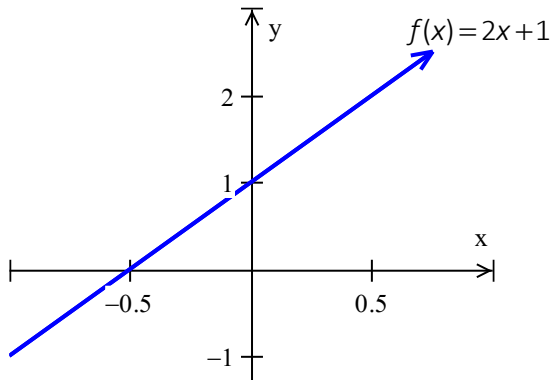
Draw the graph of $f(x) = 2x + 1$

Solution

Using x and y intercepts,

At x -intercept, $y = 0$ thus $2x + 1 = 0$ then $x = -\frac{1}{2}$

At y -intercept, $x = 0$ thus, $y = 1$

**Graphs of quadratic polynomial**

The functions of the form $f(x) = ax^2 + bx + c$ where a , b and c are constants and $a \neq 0$ is referred as quadratic polynomial.

Characteristics of quadratic function

If the coefficient of x^2 (a) is positive the function has a minimum value and it is opened upward.

If the coefficient of x^2 (a) is negative the function has a maximum value and it is open downward.

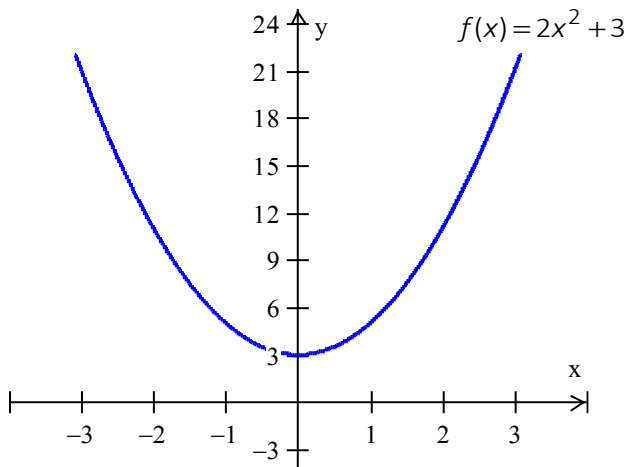
Example 3

Draw the graph of $f(x) = 2x^2 + 3$ and state domain and range of $f(x)$

Solution

Table of values

x	-3	-2	-1	0	1	2	3
y	21	11	5	3	5	11	21



Example 4

Draw the graph of $f(x) = x^2 + 2x - 1$ and $g(x) = -x^2 + 3$ on the same axes and find the values of x such that $f(x) = g(x)$

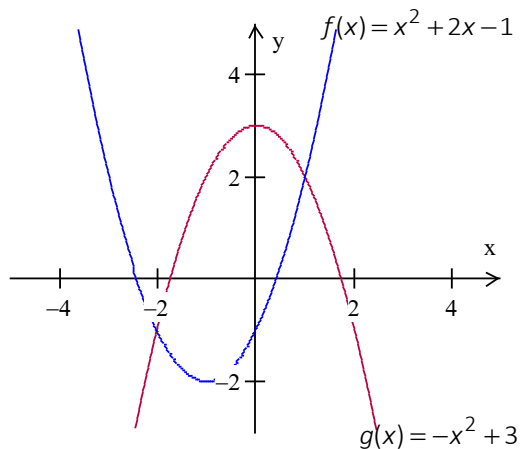
Solution

Table of value $f(x) = x^2 + 2x - 1$

x	-4	-3	-2	-1	0	1	2
y	7	2	-1	-2	-1	3	7

Table of value of $g(x) = -x^2 + 3$

x	-3	-2	-1	0	1	2	3
y	-6	-1	2	3	2	-1	-6



From the graph $f(x)=g(x)$ are the value $x=1$ and $x=-2$

Example 5

Sketch the graph of $f(x)=x^2+2x+2$ and $g(x)=x+2$, determine the values of x and y such that $f(x)=g(x)$

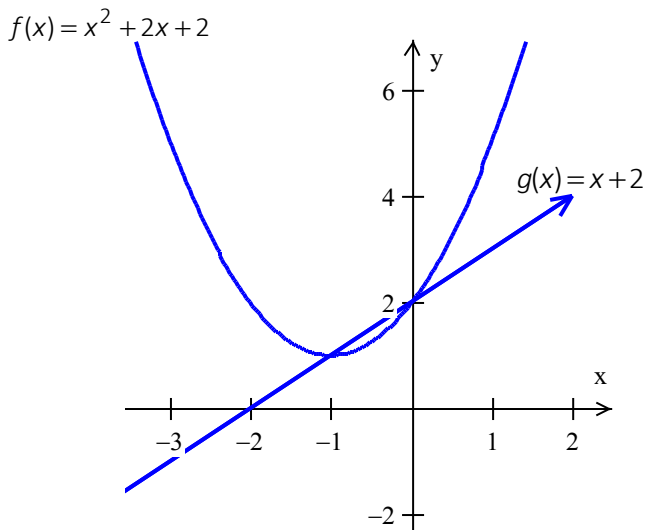
Solution

Table of values of $f(x)=x^2+2x+2$

x	-4	-3	-2	-1	0	1	2
y	10	5	2	1	2	5	10

Table of values of $g(x)=x+2$

x	-2	-1	0	1	2
y	0	1	2	3	4



From the graph the coordinates of intersection are $(-1, 1)$ and $(0, 2)$

Graphs of quadratic functions using intercepts and turning points

Consider the quadratic function $y = ax^2 + bx + c$

$$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \text{ thus } y = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$y - \frac{4ac - b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

If $a > 0$ then $a\left(x + \frac{b}{2a}\right)^2 > 0$ therefore $y \geq \frac{4ac - b^2}{4a}$ which mean the function attain its minimum value.

If $a < 0$ then $a\left(x + \frac{b}{2a}\right)^2 < 0$ therefore $y \leq \frac{4ac - b^2}{4a}$ which mean the function attain it is maximum value.

In both cases above if $a\left(x + \frac{b}{2a}\right)^2 = 0$, $x + \frac{b}{2a} = 0$ therefore $x = -\frac{b}{2a}$ is called line of symmetry.

Maximum/minimum and the line of symmetry above forms a turning point

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

Example 6

Draw the graph of $y = 2x^2 + 3x + 1$

Solution

Turning point

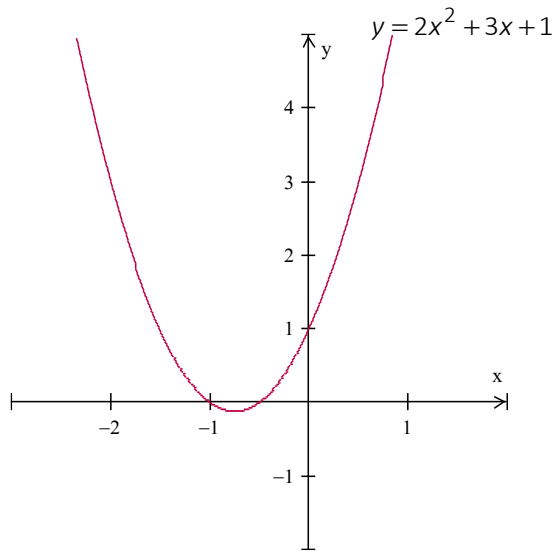
$$T\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

$$T\left(-\frac{3}{4}, -\frac{1}{4}\right)$$

y – Intercept, $x=0$ then
 $y=1$ and x – Intercept,

$$2x^2 + 3x + 1 = 0$$

$$\text{thus } x = -\frac{1}{2} \text{ or } x = -1$$



Graphs of cubic functions

Cubic functions are functions of the form $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$

Graphs of cubic functions can easily be drawn using calculus techniques by preparing the maximum and minimum or inflexion points of the curve. Here we will use the table of values method.

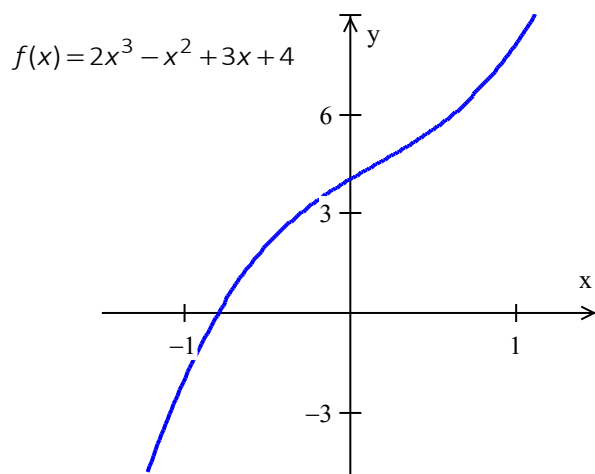
Example 7

Draw the graph of $f(x) = 2x^3 - x^2 + 3x + 4$

Solution

Table of values of $f(x) = 2x^3 - x^2 + 3x + 4$

x	-5	-4	-3	-2	-1	0	1	2	3	4
y	-286	-152	-68	-22	-2	4	8	22	58	128



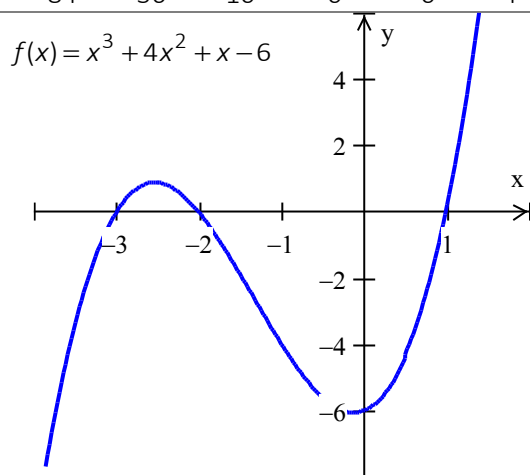
Example 8

Draw the graph of $f(x) = x^3 + 4x^2 + x - 6$

Solution

Table of values

x	-6	-5	-4	-3	-2	-1	0	1	2	3
y	-84	-36	-10	0	0	-4	-6	0	20	60



EXERCISE 1 – FUNCTIONS

1. Draw the graph of the following functions
 - (a) $y = 2x - 3$ from $-3 \leq x \leq 3$ (b) $2y = 3x - 8$
2. Draw the graph of the following
 - (a) $f(x) = 2x^2 + 3$ (b) $f(x) = x^2 - 2x$
 - (c) $f(x) = 2 - x - x^2$ (d) $f(x) = 3x - 2x^2$
3. Use all the functions in question 2 to find
 - (a) The maximum or minimum value of $f(x)$
 - (b) The line of symmetry of $f(x)$
 - (c) The turning point of $f(x)$
4. Find the value of k if the expression $2x^2 + kx + 5$ has
 - (a) Equal roots
 - (b) One root is -4
 - (c) If the roots are $-5/2$ and -1
5. Express the following quadratic function in the form of $y = a(x + b)^2 + c$
 - (a) $f(x) = 12x^2 - 3x + 5$ (b) $f(x) = x^2 - 3x$
 - (c) $f(x) = \frac{1}{3}x^2 - 10x + 2$ (d) $f(x) = 3 - 4x + 5x^2$
 - (e) $f(x) = px^2 + qx + r$
6. If $x^3 + 2x^2 + \frac{4}{3}x + b = 0$ has equal roots find the value of b .
7. Draw the graph of the following functions
 - (a) $f(x) = x^3$, $f(x) = 2x^3$ and $f(x) = 10x^3$ on the same axes, comment as the coefficient of $x^3 \rightarrow \infty$
 - (b) List 3 properties of a cubic function
 - (c) If the graph of $f(x) = 4x^3 - 3x^2 + c$ passes through the point $(1, -1)$ find the value of c and hence sketch the graph of $f(x)$, state the domain and range of $f(x)$.
8. Find the value of p from the following cubic function $f(x) = 2x^3 + px^2 + 24x + 16$ if
 - (a) It is a perfect cube
 - (b) Sketch the graph of $f(x)$ of (a) above.

9. Draw the graph of $g(x) = 3x^3 - 2x^2 + 5$ from $-2 \leq x \leq 3$, evaluate $g(\pm 50)$
10. Draw the graph of $f(x) = -2x^3 + 5x^2 - x$
11. A cubic function $f(x)$ has a slope of $x^2 - 2x + 3$ if the curve of this function $f(x)$ passes through the point $(2, -3)$, draw the graph of the function $f(x)$ and evaluate $f(3)$ and $f(-5)$

Graphs of degree four polynomials

A degree 4 polynomials is a function of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ where $a \neq 0$. These functions have the 'W' or "M" shape.

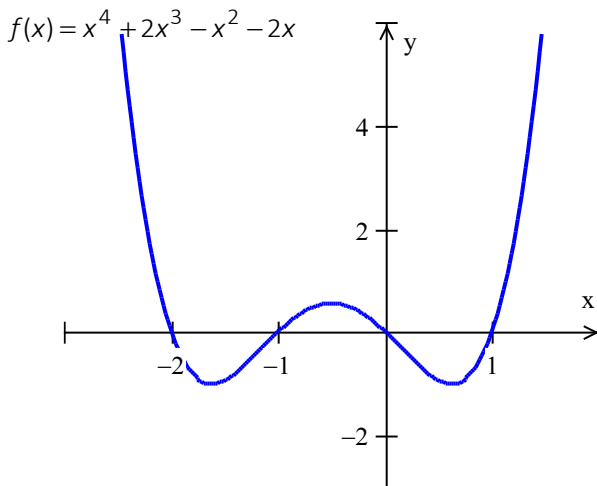
Example 9

Draw the graph of $f(x) = x^4 + 2x^3 - x^2 - 2x$

Solution

Table of values of $f(x) = x^4 + 2x^3 - x^2 - 2x$

x	-3	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	24	0	-0.9	0	0.6	0	-0.9	0	24



Example 10

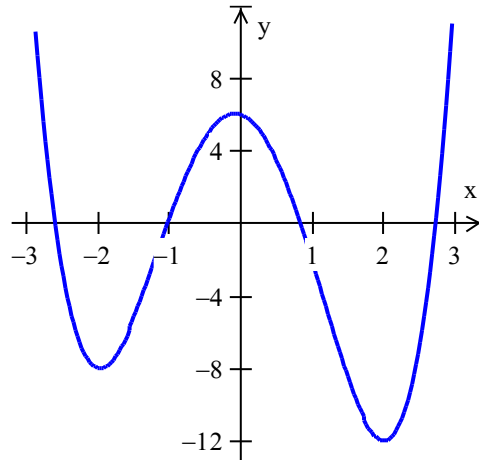
Draw the graph of $f(x) = x^4 - 8x^2 - x + 6$

Solution

Table of values

x	-3	-2	-1	0	1	2	3
y	18	-8	0	6	-2	-12	12

$$f(x) = x^4 - 8x^2 - x + 6$$

**Question**

If the function $f(x) = a(x-1) + b(x-1)(x-2)$ is divided by $x-2$ the remainder is 5 and when divided by $x-3$ the remainder is 16. Find the constants a and b . hence sketch the graph $f(x)$ and states the domain and range of $f(x)$

RATIONAL FUNCTION

A rational function is a quotient of two polynomial functions, thus $r(x) = \frac{f(x)}{g(x)}$

where $f(x)$ and $g(x)$ has no common factor(s) and $g(x) \neq 0$.

Lines called asymptotes bound graphs of rational functions; graphs never touch the asymptotes although it goes close and close to the asymptotes.

Asymptote of a curve is a line that a graph approaches but never touches the curve, is a limiting line of the curve.

There are three types of asymptotes

- (a) Vertical asymptote, $x = a$
- (b) Oblique asymptote, $y = ax + b$
- (c) Horizontal asymptote, $y = b$

There are two types of rational functions

- (a) **Proper rational function:** this is a rational function in which the numerator function has the lower degree than or equal to degree of the denominator function.

Example of proper rational function are (i) $f(x) = \frac{x+3}{x^2+3x-2}$ (ii)

$$g(x) = \frac{2x^2 + 5x - 3}{2x^2 + x - 5}$$

Proper rational function has two types' asymptotes

i. Vertical asymptote(s)

Factors in the denominator cause vertical asymptotes and/or holes.

To find:

1. Factor the denominator (and numerator, if possible).
2. Cancel common factors.
3. Denominator factors that cancel completely with a numerator factor gives a rise to a hole(s). Those that don't, give rise to vertical asymptote(s). To get the vertical asymptote, equate the denominator to zero and solve for x the value(s) of x obtained are the vertical asymptotes and they are drawn as a dotted line.

Always factorize the numerator and denominator to see if there is a hole or holes.

Example 11

Find the vertical asymptote(s) and or hole(s) of the function

$$f(x) = \frac{x^3 - 7x + 6}{x^3 + 3x^2 - 6x - 8}$$

Solution

- (a) Factorizing the denominator and numerator if possible

$$f(x) = \frac{x^3 - 7x + 6}{x^3 + 3x^2 - 6x - 8} = \frac{(x-2)(x-1)(x+3)}{(x-2)(x+1)(x+4)}$$

- (b) Cancel a common factor, $f(x) = \frac{(x-1)(x+3)}{(x+1)(x+4)}$

The factor $x-2$ is a common factor and it causes a hole, at $x=2$

- (c) The other denominator factors lead to the vertical asymptote, $(x+1)(x+4)=0$. The vertical asymptotes are $x=-1$ and $x=-4$

ii. Horizontal asymptote(s)

To get the horizontal asymptote, divide all terms of the numerator and denominator by the highest degree of x of the denominator function, simplify the terms and all other terms with $1/x^n$ (where n is a positive integer) is termed to zero.

- (a) If the numerator degree is less than the denominator degree, the

horizontal asymptote is zero. i.e., $f(x) = \frac{\alpha x^n + \beta x^k + \dots + z}{\alpha x^m + \beta x^p + \dots + \gamma}$ and if

$n < k, m > p$ and $n < m$ then the **horizontal asymptote is zero**, $f(x) = 0$

- (b) If the numerator degree equals to the denominator degree, the

horizontal asymptote is the ratio of the leading coefficients of the numerator and denominator functions. i.e.

$$f(x) = \frac{\alpha x^n + \beta x^k + \dots + z}{\alpha x^m + \beta x^p + \dots + \gamma} \quad \text{and} \quad n = m, \quad \text{then the horizontal}$$

asymptote is $f(x) = y = \frac{\alpha}{\alpha}$.

- (c) If the numerator degree is greater than the denominator degree by one, the rational function **has no horizontal asymptote** instead it has

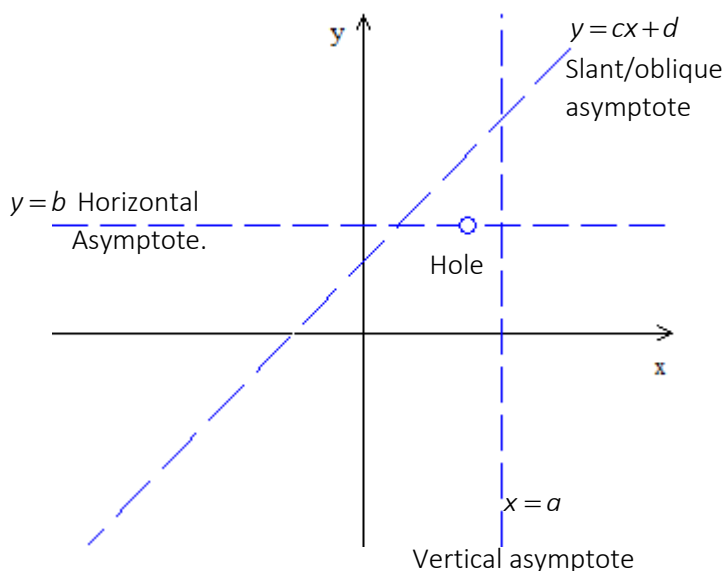
a **slant/oblique asymptote**. i.e., $f(x) = \frac{ax^n + bx^k + \dots + z}{\alpha x^m + \beta x^p + \dots + \gamma}$ and

$n > m$, by one, then the slant asymptote is a linear function, $y = ax + b$.

- (d) If the numerator degree is greater than the denominator degree by more than one degree, the rational function has a curvilinear asymptote. (This asymptote is not discussed in this book)

Note that:

When finding asymptotes always write the rational function in lowest terms. It is best not to have the function in factored form.



Example 12

Find the horizontal or slant asymptote from the following functions

(a) $f(x) = \frac{10x^2 - 3x + 2}{2x^2 - 5x + 6}$

(b) $f(x) = \frac{x+3}{x^2-4}$

$$(c) f(x) = \frac{2x^3 - 3}{x^2 - 1}$$

Solution

- (a) The numerator degree, $10x^2 - 3x + 2$ equals to the denominator degree, the horizontal asymptote exists and it is the ratio of the leading coefficients, $y = f(x) = \frac{10}{2} = 5$
- (b) The numerator degree is less than the denominator degree, therefore the horizontal asymptote is zero, $y = f(x) = 0$
- (c) The numerator degree is greater than the denominator degree, therefore horizontal asymptote doesn't exist. Instead, there is a slant asymptote, which can be obtained by dividing the numerator by the denominator using long division method or by dividing each term of the numerator by the highest degree of x of the denominator.

By long division,

$$\begin{array}{r} 2x \\ x^2 - 1 \overline{) 2x^3 + 0x^2 + 0x - 3} \\ \underline{2x^3 + 0x^2 - 2x} \\ 2x - 3 \end{array}$$

$$\text{Therefore, } f(x) = 2x + \underbrace{\frac{2x - 3}{x^2 - 1}}_{\text{ignore}}$$

$$\text{Or } f(x) = 2x + \lim_{x \rightarrow \infty} \left(\frac{2x - 3}{x^2 - 1} \right) = 2x$$

We have $y = 2x$ as the asymptote.

(b) **Improper rational function**; this is a rational function in which the numerator function has the higher degree than the denominator function. Example of improper rational function are:-

$$(i) f(x) = \frac{x^2 + 5x + 6}{x - 4} \quad (ii) g(x) = \frac{x^2 - 4}{x + 3}$$

Improper rational function has two asymptotes

- (i) **A hole:** It is possible to have holes in the graph of a rational function. Before putting the rational function into lowest terms, factor the numerator and denominator. If there is the same factor in the numerator and denominator (common factor), there is a hole. Set this factor equal to zero and solve. The solution is the x -value of the hole. Now simplify the rational function (cross out the factor that is the numerator and denominator). *Put the x -value of the hole into the simplified rational function. This will give the y -value of the hole.*
- (ii) **Oblique or slant asymptote**
To get an oblique asymptote divide the numerator function by the denominator function and the quotient obtained is the slant asymptote. Just ignore the remainder. *You should also note that a rational function will either have a horizontal or slant asymptote but not both.*

Graphs of rational function

Note that, Graphs of rational functions never cross vertical asymptotes, but may cross other types asymptote.

Example 11

Sketch the graph of $f(x) = \frac{x+2}{x+3}$ and state the domain and range.

Solution

Find the asymptotes

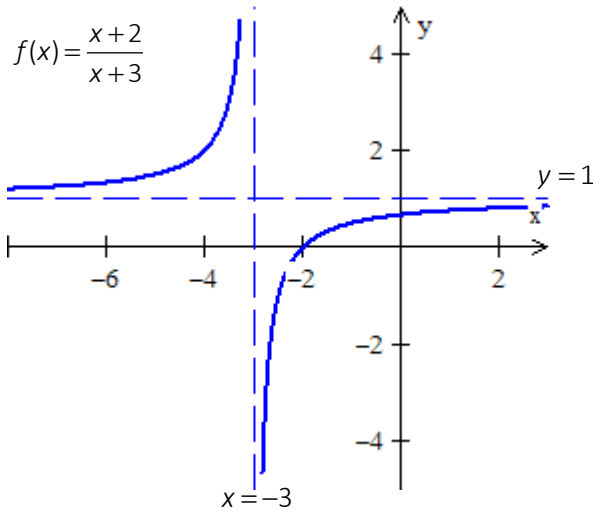
- (a) Vertical asymptote, $x+3=0 \Rightarrow x=-3$
- (b) Horizontal asymptote, $y=1$

To the right of line $x=-3$ as $x \rightarrow +\infty$, $y \rightarrow 1$

To the right of line $x=-3$ as $x \rightarrow -3$, $y \rightarrow -\infty$

To the left of line $x=-3$ as $x \rightarrow -\infty$, $y \rightarrow 1$

To the left of line $x=-3$ as $x \rightarrow -3$, $y \rightarrow +\infty$



Domain = $\{x \mid x \in \mathbb{R}, x \neq -3\}$ and range = $\{y \mid y \in \mathbb{R}, y \neq 1\}$

Example 12

Draw the graph of $y = \frac{1}{x-2}$ and state the domain and range of y .

Solution

Vertical asymptote

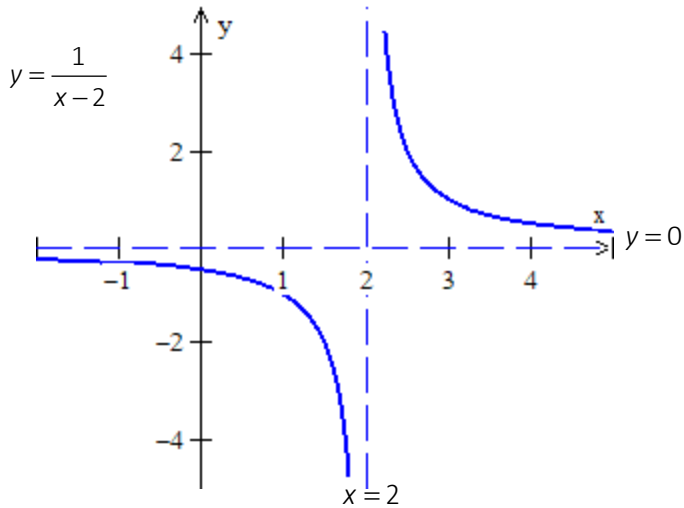
Denominator = 0, $x = 2$

Horizontal asymptote, $y = 0$

Domain = $\{x \mid x \neq 2\}$

Range = $\{y \mid y \neq 0\}$

x	-2	-1	0	1	1.99	2	2.01	3	4
y	-0.25	-0.33	-0.5	-1	-100	∞	100	1	0.5



Example 13

Sketch the graph of $y = \frac{x+2}{(x+1)(x-2)}$

Solution

Vertical asymptote, $(x+1)(x-2)=0 \Rightarrow x=-1$ and $x=2$

Horizontal asymptote, $y = \frac{x+2}{(x+1)(x-2)}$

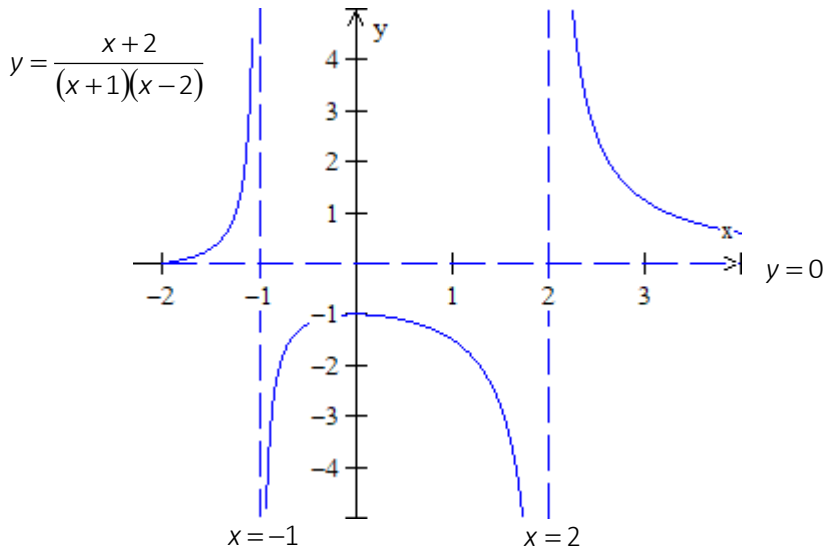
$$y = \frac{x+2}{x^2 - 2x + x - 2}$$

$$y = \frac{x+2}{x^2 - x - 2}$$

$$y = \frac{\frac{x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}$$

$$y = \lim_{x \rightarrow \infty} \left(\frac{\frac{1}{x} + \frac{2}{x^2}}{1 - \frac{1}{x} - \frac{2}{x^2}} \right)$$

The horizontal asymptote, $y=0$



Example 14

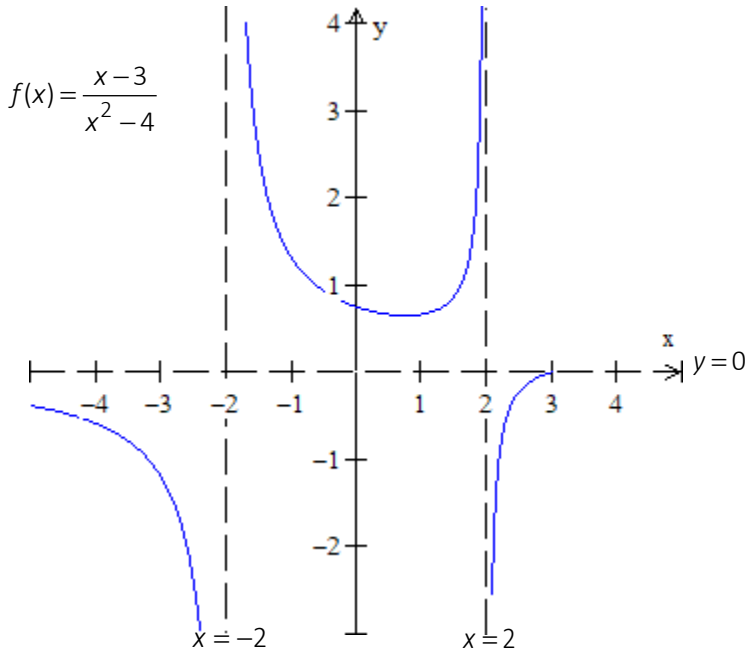
Draw the graph of $f(x) = \frac{x-3}{x^2-4}$ and state the domain and range of $f(x)$

Solution

Vertical asymptote: $x^2 - 4 = 0 \Rightarrow x = \pm 2$

Horizontal asymptote: $y = \frac{\frac{x}{x^2} - \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}}$

$y = \lim_{x \rightarrow \infty} \left(\frac{\frac{x}{x^2} - \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}} \right)$, the horizontal asymptote, $y=0$



Domain = $\{x \mid x \neq -2 \text{ and } x \neq 2\}$ and Range = $\{y \mid y \neq 0\}$

Example 15

Sketch the graph of $f(x) = \frac{x^2-2}{x+1}$ hence determine the domain and range of $f(x)$

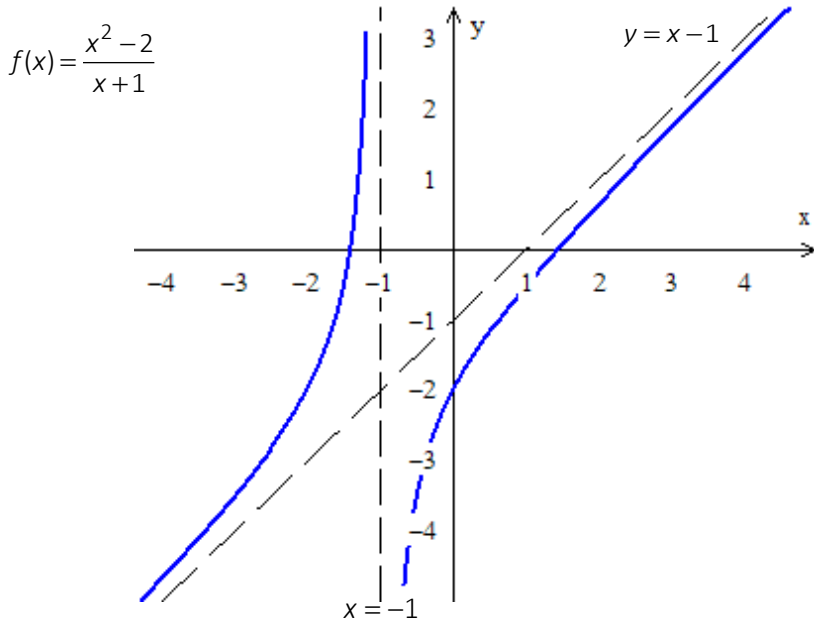
Solution

Vertical asymptote: $x+1=0 \Rightarrow x=-1$

Slant asymptote, $y = \frac{x^2-2}{x+1}$

$$\begin{array}{r}
 x-1 \\
 x+1 \overline{) x^2 + 0x - 2} \\
 \underline{x^2 + x} \\
 -x - 2 \\
 \underline{-x - 1} \\
 -1
 \end{array}$$

The slant/oblique asymptote is $y = x - 1 - \underbrace{\frac{1}{x+1}}_{\text{ignore}} \Rightarrow y = x - 1$



Example 16

Draw the graph of $f(x) = \frac{x^3 - 3x^2 + 10}{x^2 - 9}$

Solution

The vertical asymptotes $x^2 - 9 = 0$

The vertical asymptote is $x = \pm 3$

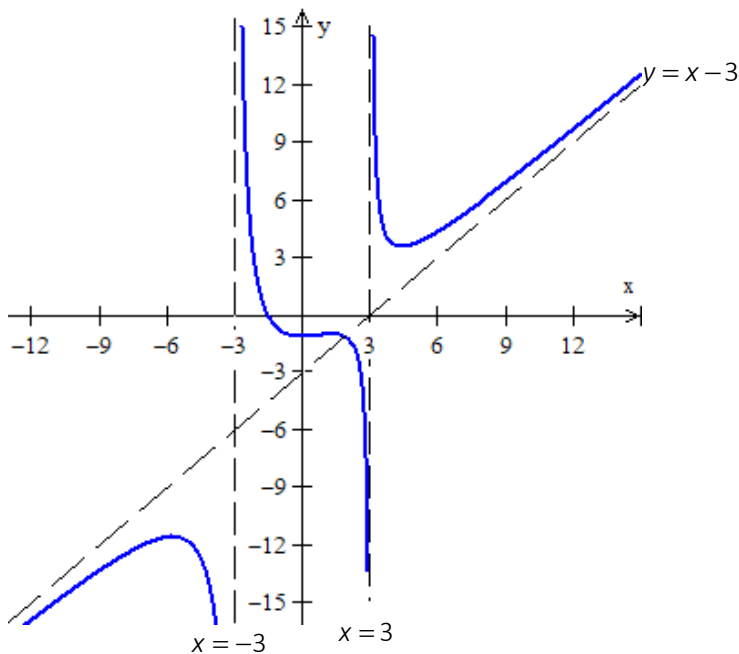
The slant asymptote by long division

$$\begin{array}{r}
 x - 3 \\
 x^2 + 0x - 9 \overline{) x^3 - 3x^2 + 0x + 10} \\
 \underline{x^3 + 0x^2 - 9x} \\
 -3x^2 + 9x + 10 \\
 \underline{-3x^2 + 0x + 27} \\
 9x - 17
 \end{array}$$

The quotient is $y = x - 3 + \underbrace{\frac{9x-17}{x^2-9}}_{\text{ignore}}$, then the slant asymptote is $y = x - 3$

This function is now into the form $y = y_1 + y_2$ where y_2 is a remainder, this remainder is ignored because as x approaches infinity, the remainder is approaching zero.

$$\text{Such that, } \lim_{x \rightarrow \infty} \left(\frac{9x-17}{x^2-9} \right) = \lim_{x \rightarrow \infty} \left(\frac{\frac{9}{x} - \frac{17}{x^2}}{1 - \frac{9}{x^2}} \right) = \frac{0}{1} = 0$$



Example 17

Draw the graph of $y = \frac{2x^2 - 3x - 2}{x^2 + x - 6}$

Solution

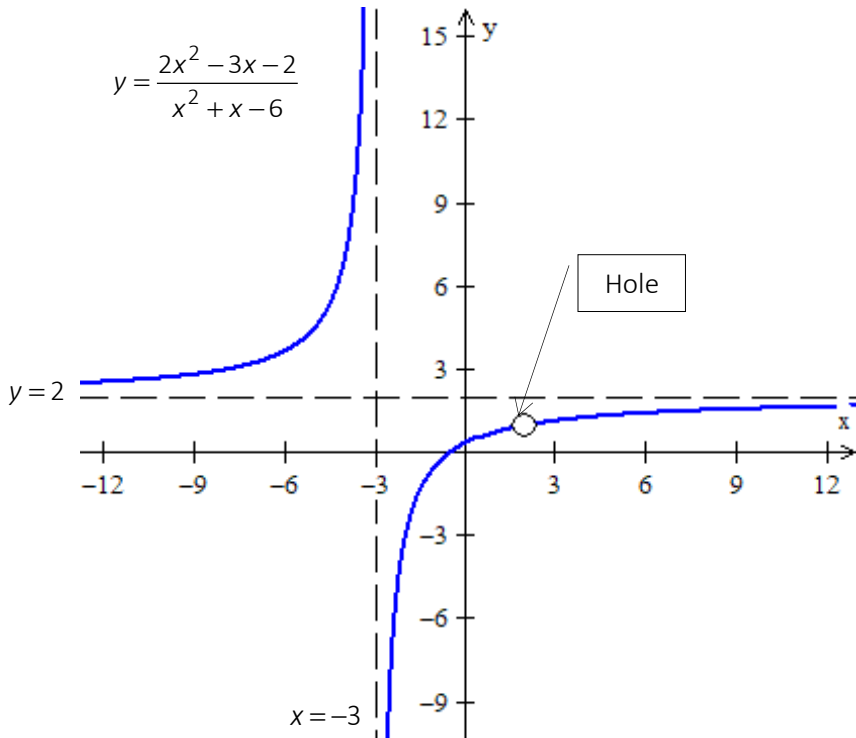
$$\text{Vertical asymptote, } y = \frac{2x^2 - 3x - 2}{x^2 + x - 6} = \frac{(2x+1)(x-2)}{(x+3)(x-2)}$$

$$(x+3)=0$$

The vertical asymptotes are $x = -3$

But $y = \frac{2x^2 - 3x - 2}{x^2 + x - 6} = \frac{(2x+1)(x-2)}{(x+3)(x-2)}$ at $x = 2$ there is a hole.

Horizontal asymptote, $y = \frac{2x+1}{x+3} = 2$



EXERCISE 2 – FUNCTIONS

1. (a) Sketch the graph of $f(x) = 3x^4 - 16x^3 + 18x^2$ state the domain and range of $f(x)$
 - (b) Given $g(x) = \frac{x^2 - 3x + 4}{x - 4}$ state for which value x the function is defined and hence sketch the graph of $g(x)$
 - (c) Determine the rational function with vertical and horizontal asymptotes 3 and 2 respectively, with x - and y -intercepts -3 and -2 respectively and hence sketch the graph of this function
2. (a) Given the function $f(x) = \frac{x-2}{x^2-4}$, determine, vertical asymptote, horizontal asymptote and or a hole if any, sketch the graph of $f(x)$ and state the domain and range of $f(x)$
 - (b) Sketch the graph of $g(x) = x^4 + x^3 - 7x^2 - x + 6$ and use differentiation to describe the stationary points of the curve
 - (c) On the same axis, draw the graph of the functions $f(x) = 2x^4 - 8x^2$ and $g(x) = -2x^4 + 8x^2$ and shade the area between the curves.
3. (a) For what condition the rational function have
 - (i) A hole (ii) A slant asymptote (iii) A horizontal asymptote $y = 0$
 - (b) Draw the graph of $f(x) = \frac{2}{(x-2)(3-x)}$
 - (c) Sketch the graph of $f(x) = \frac{x^2-4}{x-2}$ and state the domain and range of $f(x)$
4. Given $f(x) = \frac{x}{x-1}$ and $g(x) = \frac{2}{x+1}$ sketch the graph of;-
 - (a) $f(x)$ and $g(x)$ on the same axes
 - (b) $f \circ g(x)$ (c) $g \circ f(x)$
5. Define the following as used in mathematical functions
 - (a) Relation (d) A rational function
 - (b) Function (c) Asymptote

COMPOSITE FUNCTIONS

Arithmetic composition of functions

We can build up complicated functions from simple functions by using the process of composition, where the output of one function becomes the input of another. It is also sometimes necessary to carry out the reverse process, decomposing a complicated function into two or more simple functions.

Two or more functions can be added, subtracted, multiplied or divided to get a new function. Just as a number can be combined by another number using the above-mentioned operators, functions can also be combined in the same way.

If $f(x) = 3x - 5$ and $g(x) = x^2 + 3$ are two functions of x , then

- (a) $f(x) + g(x) = (3x - 5) + (x^2 + 3) = x^2 + 3x - 2$
- (b) $f(x) - g(x) = (3x - 5) - (x^2 + 3) = -x^2 + 3x - 8$
- (c) $f(x) \cdot g(x) = (3x - 5)(x^2 + 3) = 3x^3 - 5x^2 + 9x - 15$
- (d) $\frac{f(x)}{g(x)} = \frac{3x - 5}{x^2 + 3}$

Example 18

Given that $f(x) = x^2 + 3x - 4$ and $g(x) = 5 - 3x$ find

- (a) $(f + g)(x)$
- (b) $(f - g)(x)$
- (c) $\frac{g(x)}{f(x)}$

Solution

- (a) $f(x) + g(x) = x^2 + 3x - 4 + 5 - 3x$
 $f(x) + g(x) = (f + g)(x) = x^2 + 1$
- (b) $f(x) - g(x) = (f - g)(x) = (x^2 + 3x - 4) - (5 - 3x)$
 $(f - g)(x) = x^2 + 6x - 9$
- (c) $\frac{g(x)}{f(x)} = \frac{5 - 3x}{x^2 + 3x - 4}$

If $f(x)$ and $g(x)$ are functions of x , then if f compose of g is written as $f \circ g(x)$ this means the output of g is the input of f and when g

composes of f is written as $g \circ f(x)$, mean that the output of f is the input of g .

Example 19

If $f(x) = 3x + 5$ and $g(x) = \sqrt{x}$ find

- (a) $f \circ g(x)$ (b) $g \circ f(x)$ (c) $g \circ g(x)$ (d) $f \circ f(x)$

Solution

$$(a) \quad f \circ g(x) = f(g) = 3g + 5$$

$$f \circ g(x) = 3(\sqrt{x}) + 5$$

$$f \circ g(x) = 3\sqrt{x} + 5$$

$$(b) \quad g \circ f(x) = g(f) = \sqrt{f}$$

$$g \circ f(x) = \sqrt{3x + 5}$$

$$(c) \quad g \circ g(x) = g(g) = \sqrt{g}$$

$$g \circ g(x) = \sqrt{\sqrt{x}}$$

$$(d) \quad f \circ f(x) = f(f) = 3f + 5$$

$$f \circ f(x) = 3(3x + 5) + 5$$

$$f \circ f(x) = 9x + 20$$

Example 20

Given $f(x) = x^2 + 3x - 1$ and $g(x) = x - 2$ find (a) $f \circ g(x)$ (b) $g \circ f(x)$

Solution

$$(a) \quad f \circ g(x) = g^2 + 3g - 1$$

$$f \circ g(x) = (x - 2)^2 + 3(x - 2) - 1$$

$$f \circ g(x) = x^2 - x - 3$$

$$(b) \quad g \circ f(x) = f - 2$$

$$g \circ f(x) = (x^2 + 3x - 1) - 2$$

$$g \circ f(x) = x^2 + 3x - 3$$

Example 21

If $f \circ g(x) = 2x^2 + 6x + 1$ and $f(x) = 2x - 1$ find (a) $g(x)$ (b) $g \circ f(x)$

Solution

$$(a) \quad \text{From } f \circ g(x) = f(g) = 2g - 1$$

$$f \circ g(x) = 2g - 1$$

$$2x^2 + 6x + 1 = 2g - 1$$

$$2x^2 + 6x + 2 = 2g$$

$$g(x) = x^2 + 3x + 1$$

$$(b) \quad g \circ f(x) = g(f) = f^2 + 3f + 1$$

$$g \circ f(x) = (2x - 1)^2 + 3(2x - 1) + 1$$

$$g \circ f(x) = 4x^2 + 2x - 1$$

Example 22

Given $g(x) = \sqrt{x-1}$ and $g \circ f(x) = x^2 - 9$ find

$$(a) \quad f(x)$$

$$(b) \quad f \circ g(x)$$

Solution

$$(a) \quad g \circ f(x) = g(f) = \sqrt{f-1}$$

$$x^2 - 9 = \sqrt{f-1}$$

$$(x^2 - 9)^2 = f - 1$$

$$f - 1 = x^4 - 18x^2 + 81$$

$$f(x) = x^4 - 18x^2 + 82$$

$$(b) \quad f \circ g(x) = f(g) = (\sqrt{x-1})^4 - 18(\sqrt{x-1})^2 + 82$$

$$f \circ g(x) = (x-1)^2 - 18(x-1) + 82$$

$$f \circ g(x) = x^2 - 20x + 101$$

Example 23

If $f(x) = 3x + 2$, $g(x) = x - 1$ and $h(x) = x^2 - 2$ show whether $(f \circ g) \circ h(x) = f \circ (g \circ h)(x)$

Solution

Left hand side

$$f \circ g(x) = 3(x-1) + 2$$

$$f \circ g(x) = 3x - 1$$

$$(f \circ g) \circ h(x) = 3(x^2 - 2) + 2$$

$$(f \circ g) \circ h(x) = 3x^2 - 4$$

Right hand side

$$g \circ h(x) = x^2 - 1 - 2$$

$$g \circ h(x) = x^2 - 3$$

$$f \circ (g \circ h)(x) = 3(x^2 - 3) + 2$$

$$f \circ (g \circ h)(x) = 3x^2 - 7$$

From the conclusion above, function composition is not associative, thus $(f \circ g) \circ h(x) \neq f \circ (g \circ h)(x)$

GRAPH OF COMPOSITE FUNCTIONS

If we combine two or more functions the formed functions can be of any form, and the graph of the formed function can be drawn using the previous knowledge. Consider the few examples below.

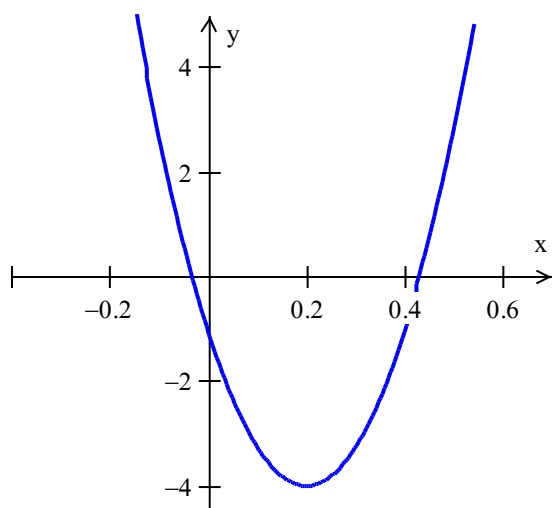
Example 24

If $f(x) = 3x^2 - 4$ and $g(x) = 5x - 1$, draw the graph of $f \circ g(x)$

Solution

$$f \circ g(x) = 3(5x - 1)^2 - 4$$

$$f \circ g(x) = 3(25x^2 - 10x + 1) - 4. \text{ Thus } f \circ g(x) = 75x^2 - 30x - 1$$



Example 25

Given that $f(x) = \frac{x}{x-1}$ draw the graph of $f \circ f(x)$ from $-3 \leq x \leq 3$

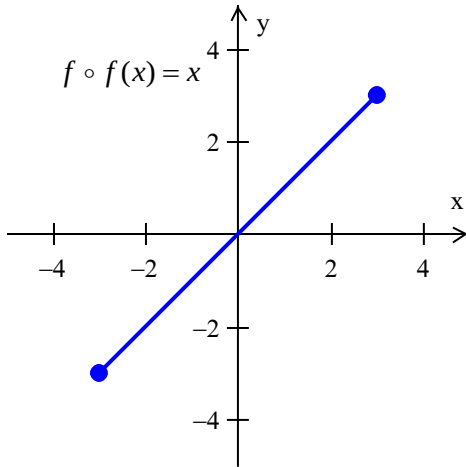
Solution

$$f(x) = \frac{x}{x-1}$$

$$f \circ f(x) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} \text{ thus } f \circ f(x) = x$$

Table of values

x	-3	-2	-1	0	1	2	3
y	-3	-2	-1	0	1	2	3



Draw the graph of $f(x) = 3 - 2x$ and $g(x) = \frac{3x}{2x - 10}$

Example 26

Solution

$$g \circ f(x) = \frac{3(3 - 2x)}{2(3 - 2x) - 10}$$

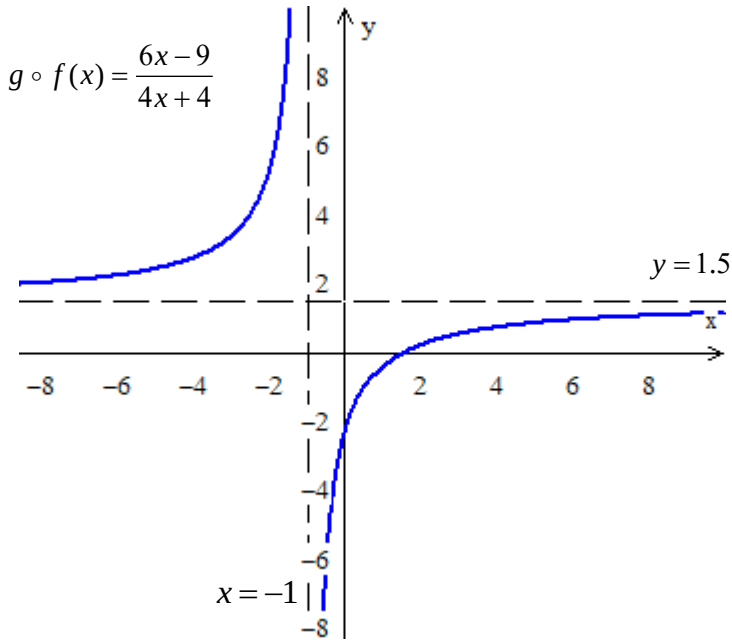
$$g \circ f(x) = \frac{6x - 9}{4x + 4}$$

Vertical asymptote

$$4x + 4 = 0 \text{ then } x = -1$$

Horizontal asymptote

$$y = \frac{6}{4} = 1\frac{1}{2}$$



Questions

- Given that $f(x) = x + 2$ and $g(x) = x^2 - 3$, find
 - $g \circ f(x)$
 - $f \circ g(x)$
 - $f^{-1} \circ g(x)$
 - $g^{-1} \circ f(x)$
- If $g(x) = 3x + 4$ and $g \circ f(x) = 3x^2 - 8$ find
 - $f(x)$
 - Domain and range of $f(x)$
 - Is $g^{-1}(f(x))$ a function?
 - If $h(x) = 2x$ show whether the composite function is associative.

EXERCISE 3 – FUNCTIONS

1. Show if $f(x) = x^2 - 4x + 3$ and $g(x) = x - 3$ find
 - (a) $(f + g)(x)$
 - (b) $(f - g)(x)$
 - (c) $f(x) \cdot g(x)$
 - (d) $\frac{f(x)}{g(x)}$
2. Given $f(x) = 2x^2 + 3x - 10$ and $g(x) = \sqrt{x - 1}$ find
 - (a) $f \circ g(x)$
 - (b) $g \circ f(x)$
3. Sketch the graph of $h(x) = x^3 - 2x^2 + 4$ and $k(x) = -x^3 + 2x - 2$ and find the intersection coordinate correct to one decimal place.
4. A function $f(x) = \frac{ax + b}{kx + 3}$ has vertical asymptote $x = -3$ and horizontal asymptote $y = 1$ if the x-intercept is 4 find the value of a , b and k .
5. If $g(x) = 2x^2 - 3x + 1$ and $f(x) = x^2 + 4$ find
 - (a) $f \circ g(-5)$
 - (b) $g \circ f(0)$
6. (i) Given $f(x) = x - 1$ and $g(x) = x^2 + 2x - 8$ find
 - (a) $f \circ g(x)$
 - (b) $g \circ f(x)$
 (ii) Given $f(x) = 2x - k$ and $g(x) = 3x + 1$ find the value of k if $f \circ g(x) = g \circ f(x)$
7. Given $f(x) = \begin{cases} -2x^2 + 1 & \text{if } x > -1 \\ x - 2 & \text{if } x \leq -1 \end{cases}$ and $g(x) = x + 2$ find $f \circ g(x)$ and sketch the graph of $f \circ g(x)$
8. The domain of $g(x) = 2x - 1$ is $\{x : -1, 3, 4, 7, 8, 10\}$ find the domain and range of $f \circ g(x)$ if $f(x) = x + 1$

5.4 GRAPHS OF THE EXPONENTIAL FUNCTIONS

The functions of the form $f(x) = b^x$ where b is a constant called a *base*, where it can be $b > 1$ or $b < 1$ and x is a variable or another function is called an exponential function.

Exponential functions can be used in many contexts, such as compound interest (money), population growth and radioactive decay.

Properties of the exponential function, $f(x) = b^x$

- (a) The graph is asymptotic to x-axis, i.e. $y = f(x) = 0$
- (b) If $b > 1$ the function is an increasing function
- (c) If $0 < b < 1$ the function is a decreasing function
- (d) The function is a continuous
- (e) It passes the point $(0, 1)$
- (f) Domain = $\{x : x \text{ is set of all real numbers}\}$
- (g) Range = $\{y : y > 0\}$
- (h) The graph is smooth

Example 27

If $f(x) = 2^x$ sketch the graph of $f(x)$ and state the domain and range of $f(x)$.

Solution

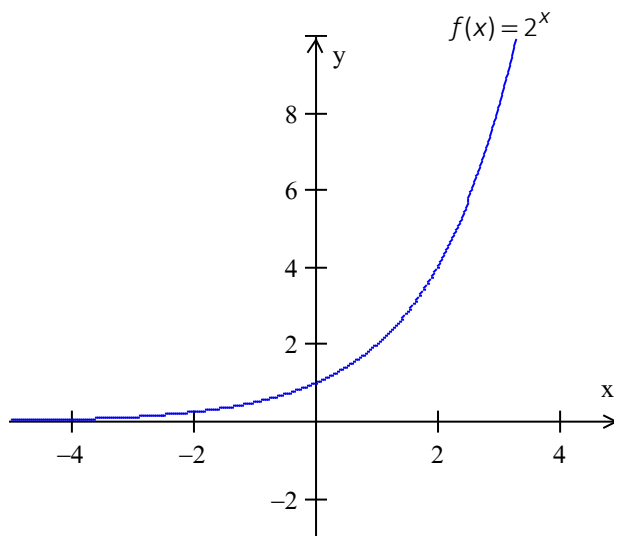
Table of values

x	-3	-2	-1	0	1	2	3
y	0.125	0.25	0.5	1	2	4	8

From the graph below

Domain = $\{x : x \in \text{all real numbers}\}$

Range = $\{y : y > 0\}$



Example 28

Draw the graph of $y = 3^x$ and $y = 3^{-x}$ on the same axis

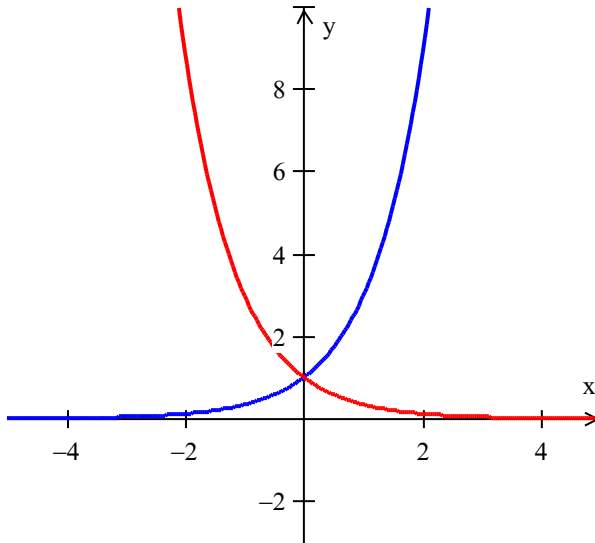
Solution

Table of values

x	-3	-2	-1	0	1	2	3
$y = 3^x$	0.04	0.11	0.33	1.00	3.00	9.00	27.00
$y = 3^{-x}$	27.00	9.00	3.00	1.00	0.33	0.11	0.04

$$y = 3^{-x}$$

$$y = 3^x$$



General exponential function

These are exponential functions of the form $f(x) = ka^x$ where k is any non-zero real number

Properties of these functions, it passes through the point $(0, k)$, and all other properties mentioned above holds.

Let consider the exponential function (e) , i.e. $f(x) = ae^x$

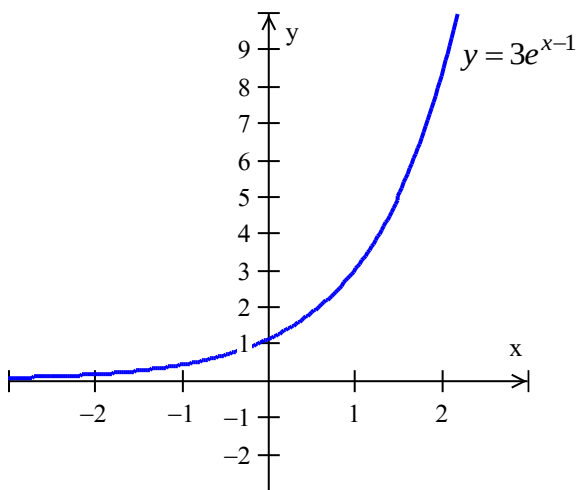
Example 29

Draw the graph of $f(x) = 3e^{x-1}$

Solution

Table of values

x	-2	-1	0	1	2
y	0.1494	0.4060	1.1036	3.0000	8.1548



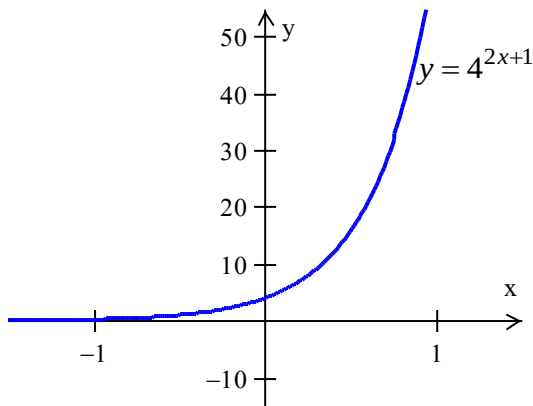
Example 30

Draw the graph of $y = 4^{2x+1}$

Solution

Table of values

x	-2	-1	0	1	2
y	0.015625	0.25	4	64	1024



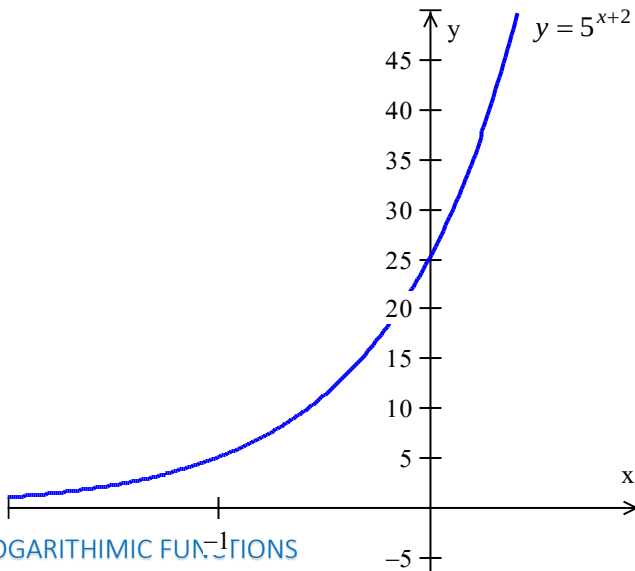
Example 31

Sketch the graph of $y = 5^{x+2}$

Solution

Table of values $y = 5^{x+2}$

x	-0.6	-0.4	-0.2	0	1	2	3
y	9.5	13.1	18.1	25	125	625	3125



5.5 LOGARITHMIC FUNCTIONS

The function of the form $y = \log_b f(x)$ or $y = \ln f(x)$ is called logarithmic function where $f(x)$ is a function of x and b is a base of the logarithm which is greater than zero. The word “log” is the short form of logarithm. Logarithm to base e is called natural logarithm abbreviated as $\ln$. Note that $\log_e x = \ln x$.

For the function $f(x) = \log x$, range of the logarithmic function is a set of all real numbers while the domain is a set of all positive real numbers.

Example 32

Draw the graph of $f(x) = \log x$

Solution

Table of values

x	0.7	0.8	0.9	1	1.1	1.2
y	-0.155	-0.097	-0.046	0.000	0.041	0.079

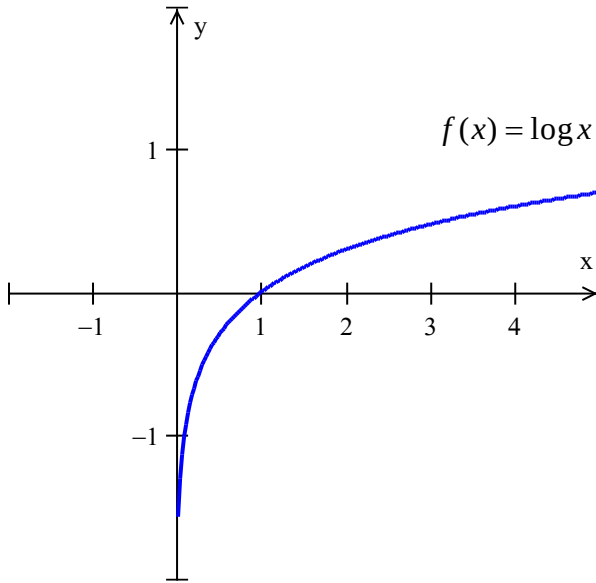
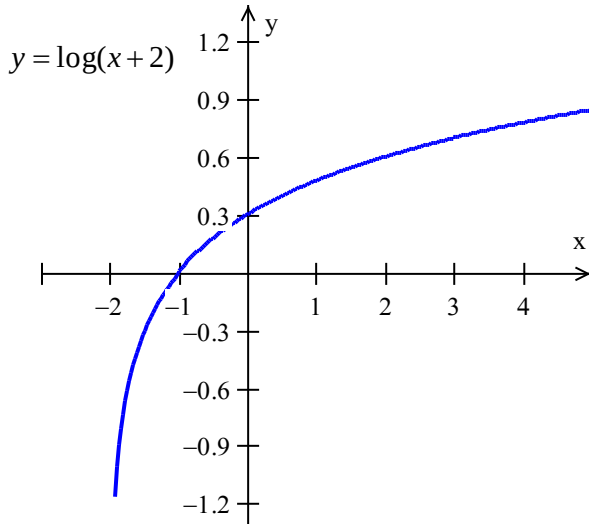
**Example 33**Draw the graph of $y = \log(x + 2)$ and state the domain and range of y .**Solution**

Table of values

x	-1.1	-0.7	-0.3	0	0.5	0.9	1.3
y	-0.05	0.11	0.23	0.30	0.40	0.46	0.52



Domain

Range

$x + 2 > 0$ therefore, $x > -2$ Make y the subject

Domain = $\{x : x > -2\}$ $y = \log(x + 2)$

$$10^y = x + 2 \Rightarrow 10^y - 2 = x$$

Range = $\{y : y \in \text{all real numbers}\}$ **Example 34**

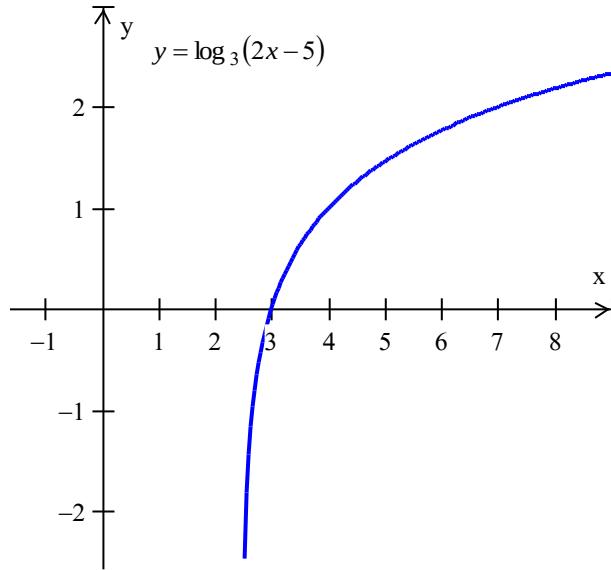
Draw the graph of $f(x) = \log_3(2x - 5)$

- (a) The asymptote of $f(x)$
 (b) State the domain and range of $f(x)$

Solution

Table of values

x	3	3.5	4	4.5	5	5.5
y	0.000	0.631	1.000	1.262	1.465	1.631



- (a) The asymptote, $y = \log_3(2x - 5)$

Logarithm of a number is defined when the number is positive

Solve for x : $2x - 5 = 0$

$$2x = 5 \Rightarrow x = 2.5$$

The vertical asymptote is $x = 2.5$

- (b) Domain = $\{x : x > 2.5\}$ and

Range = $\{y : y \text{ is a set of all real numbers}\}$

Example 35

Draw the graph of $f(x) = \ln\left(\frac{1}{2x - 4}\right)$

- (a) Find the asymptote of $f(x)$

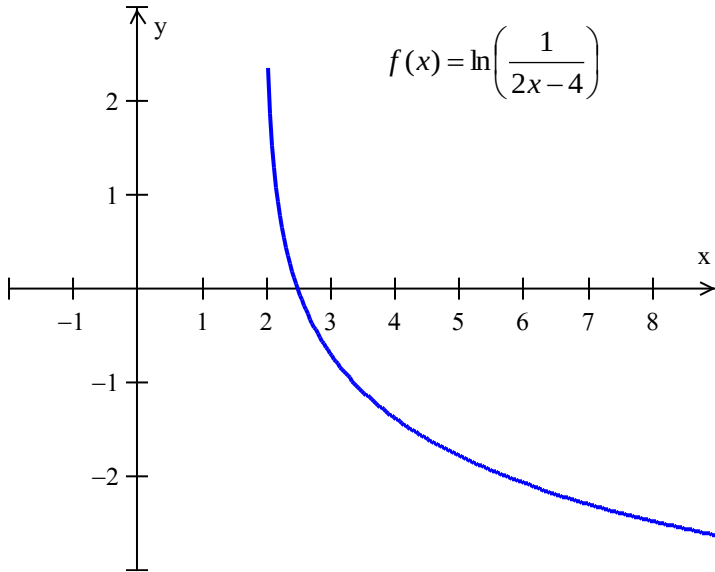
- (b) State the domain and range of $f(x)$

Solution

Table of values

$$f(x) = \ln\left(\frac{1}{2x - 4}\right) = \ln(2x - 4)^{-1}$$

$$f(x) = -\ln(2x - 4)$$



- (a) Asymptote, $2x - 4 = 0$ therefore, the vertical asymptote is $x = 2$
 (b) Domain = $\{x : x > 2\}$ and range = $\{y : y \in \mathbb{R}\}$

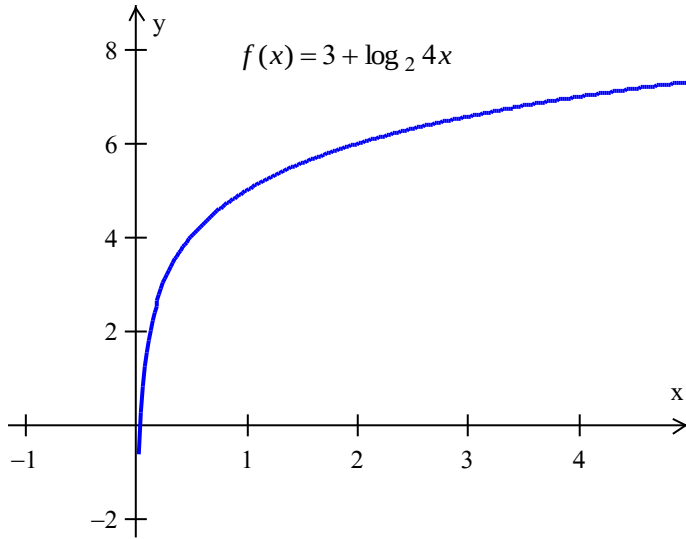
Example 36

Draw the graph of $f(x) = 3 + \log_2 4x$

Solution

Table of values

x	0.1	1	2	3	4	5
y	1.68	5.00	6.00	6.58	7.00	7.32



5.6 STEP FUNCTIONS / PIECEWISE FUNCTIONS

The functions discussed above are functions represented by a single equation. It is possible sometime to have a function with several equations with different set of domain. A function of several functions is called piecewise function or step function.

Example of piecewise function
$$f(x) = \begin{cases} y_1 & \text{if } x \geq a \\ y_2 & \text{if } b < x \leq c \\ y_3 & \text{if } x < d \end{cases}$$

Use the symbol $\circ$ (an open circle) to represent $>$ or $<$ and $\bullet$ (closed circle) use the symbol to represent $\leq$ or $\geq$.

Example 37

Draw the graph of
$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 2 \\ x + 2 & \text{if } x \geq 3 \end{cases}$$

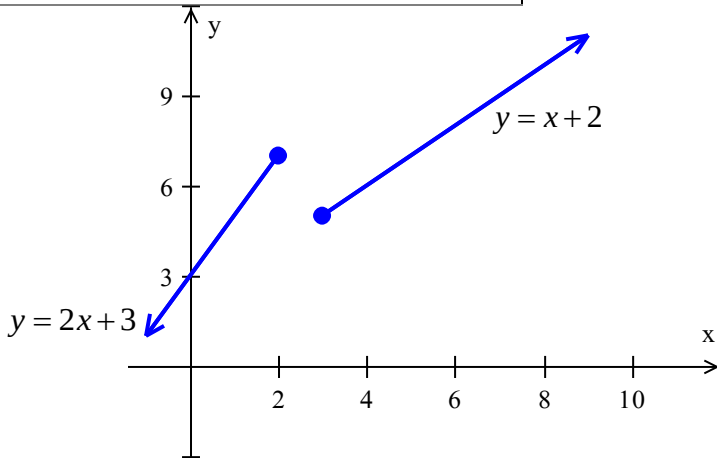
- State the domain and range of $f(x)$
- For which values of x the function is not defined
- Evaluate $f(1)$, $f(0)$ and $f(4)$

Solution

Table of values

x	3	4	5	6	7	8	9
$y = x + 2$	5	6	7	8	9	10	11

x	2	1	0	-1	-2	-3	-4
$y = 2x + 3$	7	5	3	1	-1	-3	-5



- (a) Domain = $\{x : x < 2 \text{ and } x \geq 3\}$ and Range = $\{y : y \in \mathbb{R}\}$
 (b) The function is not defined at $2 \leq x < 3$
 (c) $f(1) = 5$, $f(1) = 5$ and $f(4) = 6$

Example 38

$$\text{Given } f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ 4 & \text{if } -1 \leq x < 2 \\ x - 1 & \text{if } x \geq 2 \end{cases}$$

- (a) Draw the graph of $f(x)$
 (b) State the domain and range of $f(x)$
 (c) Evaluate $f(-10) + f(0) + f(10)$

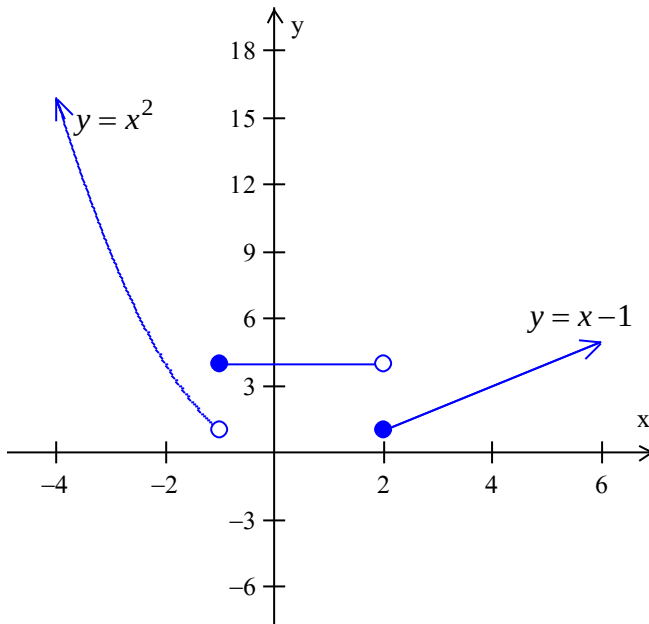
Solution

(a) Table of values

x	-1	-2	-3	-4	-5	-6
$y = x^2$	1	4	9	16	25	36

x	-1	0	1	2
$y = 4$	4	4	4	4

x	2	3	4	5	6	7
$y = x - 1$	1	2	3	4	5	6

(b) Domain = $\{x : x \in \mathbb{R}\}$ and Range = $\{y : y \geq 1\}$ (c) $f(-10) + f(0) + f(10) = 100 + 4 + 9 = 113$ **Example 39**

$$\text{Given } f(x) = \begin{cases} x & \text{if } x \leq -2 \\ -x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } 2 \leq x < 6 \\ -2 & \text{if } x \geq 6 \end{cases}$$

- (a) Draw the graph of $f(x)$
 (b) Find the domain and range $f(x)$
 (c) Evaluate $-f(-4) - f(1) - f(3) - f(100)$

Solution

- (a) Table of values

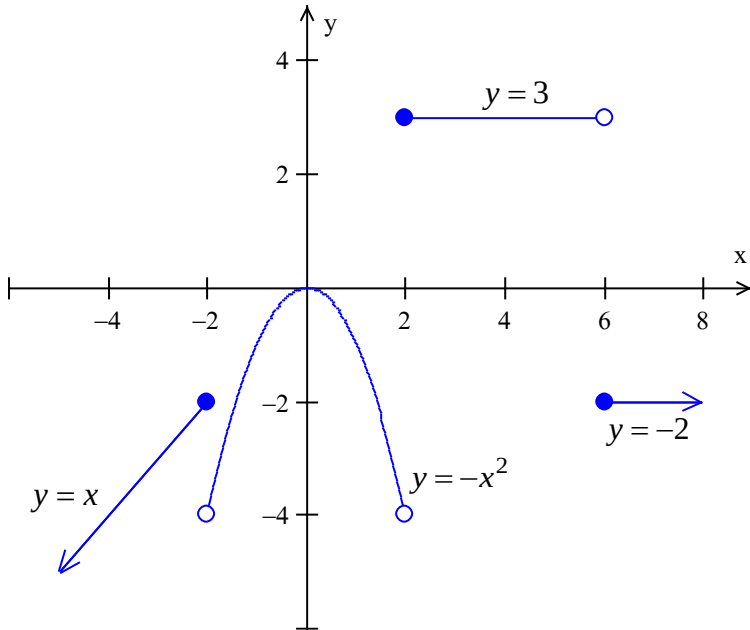
x	-2	-3	-4	-5	-6
$y = x$	-2	-3	-4	-5	-6

x	-2	-1	0	1	2
$y = -x^2$	-4	-1	0	-1	-4

x	2	3	4	5	6
$y = 3$	3	3	3	3	3

x	6	7	8	9	10
$y = -2$	-2	-2	-2	-2	-2

- (b) Domain = $\{x : x \in \mathfrak{R}\}$
 Range = $\{y : y = 3, y \geq 0\}$
 (c) $-f(-4) - f(1) - f(3) - f(100) = 4 + 1 - 3 + 2 = 4$



Questions:

- Sketch the graph of $f(x) = \frac{1}{2-x}$ and state whether $f(x)$ is a one to one function or not
- Given $f(x) = x^2 + ax + b$ show that if the equation $f(x+a) = a$ has no real roots then $a^2 < 4(b-a)$
- Given $h(x) = 28x^2 + 25x + 3$ if $h(x)$ is a product of two linear functions, find these functions and hence show whether composite of functions is commutative or not.
- Use factorization methods, solve the following quadratic equation $(a+b)^2 x^2 + 6(a^2 - b^2)x + 9(a-b)^2 = 0$

MISCELLANEOUS EXERCISE – FUNCTIONS

1. (a) Draw the graph of $f(x) = 2^{x-3}$ and state the domain and range of $f(x)$
 (b) Given $f(x) = e^x$ on the same axes, draw the graph of $f(x)$ and $f^{-1}(x)$
 (c) If $g(x) = 2x$ and $f(x) = 10^x$ draw the graph of $f(g(x))$ and state the domain and range of $f(g(x))$
2. Find the asymptote of the function $g(x) = 3\ln(2x-6)$ and sketch the graph of $g(x)$ and $g^{-1}(x)$ on the same axes.
3. The function f is defined by $f: x \rightarrow ax+b$ for $x \in \mathfrak{R}$ where a and b are constants, it is given that $f(2)=1$ and $f(5)=7$, find
 - (a) Find the values of a and b .
 - (b) Solve the equation $f(f(x))=0$
4. (a) Express $f(x) = 8x - x^2$ in the form of $a^2 - (x+b)^2$, state the numerical values of a and b
 (b) State the coordinate of the stationary point of the curve.
 (c) Find the set of values of x for which $y \geq -20$
 (d) State the domain and range of $f^{-1}(x)$
5. The function is defined as $f(x) = \begin{cases} 1 - \frac{1}{2}x & 0 \leq x \leq 2 \\ 3 & \text{otherwise} \end{cases}$
 - (a) Draw the graph of $f(x)$
 - (b) State the domain and range of $f(x)$
 - (c) $f(1) + f(-10)$
 - (d) Is $f(x)$ an injective function? Why?
6. (a) Sketch the graph of $\frac{3x}{(x-1)(x+2)}$ hence solve $(x-1)(x+2)(x+3) = 3x$
 (b) The function f , where $f(x) = a \sin x + b$, is defined for the domain $0 \leq x \leq 2\pi$. Given that $f\left(\frac{1}{2}\pi\right) = 2$ and that $f\left(\frac{3}{2}\pi\right) = -8$,

- i. Find the value of a and b
 - ii. Find the values of x for which $f(x) = 0$, giving your answer to 2 decimal places
 - iii. Sketch the graph of $y = f(x)$
7. Sketch the curve $y = \frac{x+1}{x-1}$, State the domain and range
8. (a) The functions f and g are defined by $f(x) = 3x + 2$ and $g(x) = \frac{5}{2x+4}$ sketch the graph of f and g on the same axes and hence find the values of x for which $f(x) = g(x)$ to 2 decimal places.
 (b) Solve the equation $f(x) \times g(x) = 3$
9. (a) State the properties of the function $f(x) = 5b^x$
 (b) State the properties of the function $f(x) = \log_a x$
 (c) State the properties of the function $f(x) = \ln x$
 (d) State the relation between $f(x) = b^x$ and $g(x) = \frac{\ln x}{\ln b}$ where b is constant.
10. (a) Given that the function $f(x) = \frac{3x+p}{x-r}$ is even function, show that $p + 3r = 0$.
 (b) If $f(x) = 4x^2 - x + 5$, find $g(x)$ for which $f \circ g(x) = 4x^2 - 16x + 21$
 (c) Sketch the graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$ and state the domain and range of $f(x)$.
 (d) The function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and $g(x) = \ln \sqrt{\frac{1+x}{1-x}}$ show that $f \circ g(x) = g \circ f(x)$ and comment the relation of $f(x)$ and $g(x)$
11. If $f(x) = x^2 + 3$ and $g(x) = \sqrt{x-3}$. Find $g \circ f(2)$

ADVANCED MATHEMATICS



BARAKA
L01BANGUT1