

ADVANCED MATHEMATICS

ALGEBRA

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To Muhammad ibn Musa al-Khwarizmi
"The father of Algebra"

ALGEBRA

Chapter

6

Muhammad ibn Musa al-Khwarizmi was a 9th century Muslim mathematician and astronomer. He is known as the “father of algebra”, a word derived from the title of his book, *Kitab al-Jabr*.

The end of the 19th and the beginning of the 20th century saw a tremendous shift in the methodology of mathematics. Abstract (modern) algebra emerged around the start of the 20th century, under the name **modern algebra**. Its study was part of the drive for more intellectual rigor in mathematics. Initially, the assumptions in classical algebra, on which the whole of mathematics (and major parts of the natural sciences) depend, took the form of axiomatic systems. No longer satisfied with establishing properties of concrete objects, mathematicians started to turn their attention to general theory. Formal definitions of certain algebraic structures began to emerge in the 19th century. For example, results about various groups of permutations came to be seen as instances of general theorems that concern a general notion of an abstract group. Questions of structure and classification of various mathematical objects came to forefront.

6.1 Indices

Indices is a plural of the word index. An index is an exponent. We recall that a power is the product of a certain number of factors, all of which are the same. For example, 3^7 is a power, in which the number 3 is called the base and the number 7 is called the index or exponent.

Laws of exponents

- (a) To multiply powers with the same base, add the indices, $a^m \times a^n = a^{m+n}$
- (b) To divide powers with the same base, subtract the indices, $\frac{a^m}{a^n} = a^{m-n}$
- (c) To raise a power to a power, multiply the indices, $(a^m)^n = a^{mn}$
- (d) A power of a product is the product of the powers, $a^m \times b^m = (ab)^m$

(e) A power of a quotient is the quotient of the powers, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ where $b \neq 0$

(f) Reciprocal of rational base and the effect on an exponent, $\left(\frac{a}{b}\right)^c = \left(\frac{b}{a}\right)^{-c}$

(g) Rational exponents lead to radical sign, $a^{\frac{1}{n}} = \sqrt[n]{a}$

(h) Negative exponents lead to reciprocal of a number, $a^{-4} = \frac{1}{a^4}$ and

$$b^{-\frac{1}{3}} = \frac{1}{b^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{b}}$$

(i) Zero exponent. Any number (except zero) exponent zero is one (1), $a^0 = 1$ where $a \neq 0$

Example 1

Simplify the following

$$(a) \frac{27^{\frac{1}{3}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}} \quad (b) \frac{a^{x+\frac{1}{2}y} \times b^{2x-y}}{(ab^2)^x \sqrt{a^y}} \quad (c) \frac{\left(2\frac{1}{4}\right)^{\frac{5}{2}} \times \left(\frac{8}{27}\right)^{-\frac{1}{3}}}{\left(\frac{16}{81}\right)^{\frac{1}{4}} \times \left(\frac{4}{9}\right)^{-\frac{1}{2}}}$$

Solution

$$(a) \frac{27^{\frac{1}{3}} \times 243^{\frac{1}{2}}}{243^{\frac{4}{5}}} \text{ using laws of indices}$$

$$\frac{(3^3)^{\frac{1}{3}} \times (3^5)^{\frac{1}{2}}}{(3^5)^{\frac{4}{5}}} = \left(\frac{3^{3 \times \frac{1}{3}} \times 3^{5 \times \frac{1}{2}}}{3^{5 \times \frac{4}{5}}} \right) = \left(\frac{3 \times 3^{\frac{5}{2}}}{3^{\frac{25}{4}}} \right) = \left(\frac{3^{\frac{7}{2}}}{3^{\frac{25}{4}}} \right) = \left(3^{\frac{7}{2} - \frac{25}{4}} \right) = (3)^{-11/4}$$

$$\text{Therefore } \frac{27^{\frac{1}{3}} \times 243^{\frac{1}{2}}}{243^{\frac{5}{4}}} = 3^{-11/4}$$

$$\begin{aligned} \text{(b) } & \frac{a^{x+\frac{1}{2}y} \times b^{2x-y}}{(ab^2)^x \sqrt{a^y}} \\ & \frac{a^{\left(x+\frac{1}{2}y\right)} \times b^{(2x-y)}}{(ab^2)^x \sqrt{a^y}} = \frac{a^x \times a^{\frac{1}{2}y} \times b^{2x} \times b^{-y}}{a^x \times b^{2x} \times a^{\frac{1}{2}y}} \\ & = \left(\frac{a^x}{a^x}\right) \left(\frac{a^{\frac{1}{2}y}}{a^{\frac{1}{2}y}}\right) \left(\frac{b^{2x}}{b^{2x}}\right) (b^{-y}) = \frac{1}{b^y} \end{aligned}$$

$$\begin{aligned} \text{(c) } & \frac{\left(2\frac{1}{4}\right)^{\frac{5}{2}} \times \left(\frac{8}{27}\right)^{-\frac{1}{3}}}{\left(\frac{16}{81}\right)^{\frac{1}{4}} \times \left(\frac{4}{9}\right)^{-\frac{1}{2}}} \\ & \frac{\left(\frac{9}{4}\right)^{\frac{5}{2}} \times \left(\frac{8}{27}\right)^{-\frac{1}{3}}}{\left(\frac{16}{81}\right)^{\frac{1}{4}} \times \left(\frac{4}{9}\right)^{-\frac{1}{2}}} = \frac{\left(\frac{9}{4}\right)^{\left(\frac{5}{2}-\frac{1}{2}\right)} \times \left(\frac{2}{3}\right)^{-1}}{\left(\frac{2}{3}\right)} = \left(\frac{3}{2}\right)^4 \times \left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^6 \end{aligned}$$

Solving exponential questions

$$\text{(a) If } a^x = a^y \text{ then } x = y \quad \text{(b) If } a^m = b^m \text{ then } a = b$$

Example 2

Solve for x if $2^{x+4} = 64$

Solution

Express 64 in power form, to a base of 2 if possible $64 = 2^5$

$$\therefore 2^{x+4} = 2^5 \text{ from the rule } a^x = a^y \Rightarrow x = y$$

Then $x+4=6$ therefore $x=2$

Example 3

$$\text{Solve for } x: 9^{2x-5} = 27$$

Solution

The two bases given 9 and 27 can be expressed in base 3

$$9 = 3^2 \text{ and } 27 = 3^3$$

$$\text{Then } (3^2)^{2x-5} = 3^3$$

$$3^{2(2x-5)} = 3^3$$

$$3^{4x-10} = 3^3$$

$$\text{Therefore, } x = \frac{13}{4}$$

Example 4

$$\text{Solve for } x: 3(3^{2x}) - 10(3^x) + 3 = 0$$

Solution

$$\text{Given } 3(3^{2x}) - 10(3^x) + 3 = 0$$

The equation is a quadratic equation of 3^x

Let $m = 3^x$ the equation

$$3(3^{2x}) - 10(3^x) + 3 \Rightarrow 3(3^x)^2 - 10(3^x) + 3 = 0$$

$$\text{Substitute } m: 3(3^x)^2 - 10(3^x) + 3 \Rightarrow 3m^2 - 10m + 3 = 0$$

$$\text{Solve for } m \text{ by factorization, } 3m^2 - 10m + 3 = (m-3)(3m-1) = 0$$

$$m-3=0 \Rightarrow m=3, \text{ thus } 3^x = 3 \Rightarrow x=1$$

$$\text{Also } 3m-1=0 \Rightarrow m=\frac{1}{3} \text{ but } m=3^x \text{ from } 3^x = \frac{1}{3} = 3^{-1} \text{ thus } x=-1$$

Therefore, $x = \pm 1$

Question

The variables x , and y satisfy the equation $x^n y = C$, where n and C are constants. When $x=1.10$, $y=5.20$ and $x=3.20$, $y=1.05$ find the values of n and C correct to 1 decimal place.

6.2 Logarithm

The reader should be familiar with what is logarithm, but simplify logarithm is another way of writing exponents. We had a time to solve some exponential equation above, but under one condition, that both exponents must have the same base. It is possible and easy to solve $3^{x-1} = 27$ or $8^{2x+1} = 512^{x-10}$ just because 27 and 512 can be written in a same base as 3 and 8, for the case $3^x = 2$ or $6^{2x+3} = 5^{x-2}$ is not possible without logarithm, because you cannot express 3 as 2's or 6 as 5's. Due to this logarithm is important.

Two types of logarithm

- (a) Common Logarithm (Base 10 logarithm)
- (b) Natural Logarithm (Base e logarithm)

Laws of logarithm

If $a = b^x \Rightarrow \log_b a = x$

- (a) Logarithm of power, $\log_b a^n = n \log_b a$ where b is a base and n is an exponent
- (b) If $a = b$ then $\log_b a = \log_a a = 1$
- (c) $\log_a (b \times c) = \log_a b + \log_a c$

Proof:

Let $b = m^x$ and $c = m^y$

$$b \times c = m^x \times m^y = m^{(x+y)}$$

Introduce logarithm to base a both sides

$$\log_a (b \times c) = \log_a m^{(x+y)} = x \log_a m + y \log_a m$$

$$\log_a (b \times c) = \log_a b + \log_a c \text{ Hence shown}$$

- (d) Quotient of logarithm: $\log_a \left(\frac{b}{c} \right) = \log_a b - \log_a c$

- (e) Logarithm validity, $\log_a b \in R$ for $a > 0$ and $b > 0$

(f) Change of base: $\log_a b = \frac{1}{\log_b a}$ and $\log_b a = \frac{\log_c a}{\log_c b}$

(g) If $\log_b A = \log_b C$ then $A = C$

Natural logarithm

Natural logarithm is abbreviated as $\ln$. This is a logarithm to base e . $\log_e a = \ln a$ for all $a > 0$. Since $\ln a$ is also logarithm, then all laws of logarithm holds. Note that these rules of logarithm are true only if base are the same.

Example 5

Use the properties of logarithm to re-write each expression as a single logarithm

(a) $2\log_b x + \frac{1}{2}\log_b(x+4)$

(b) $3\log(x-2) - \log(2x+4)$

(c) $4\log_a(x+2) + 3\log_a(x-5) + 2\log_a(2x+1)$

Solution

(a) $2\log_b x + \frac{1}{2}\log_b(x+4)$

$$2\log_b x + \frac{1}{2}\log_b(x+4) = \log_b x^2 + \log_b \sqrt{x+4}$$

$$2\log_b x + \frac{1}{2}\log_b(x+4) = \log_b(x^2 \sqrt{x+4})$$

(b) $3\log(x-2) - \log(2x+4) = \log \frac{(x-2)^3}{2x+4}$

(c) $4\log_a(x+2) + 3\log_a(x-5) + 2\log_a(2x+1)$

$$\log_a(x+2)^4 + \log_a(x-5)^3 + \log_a(2x+1)^2$$

$$= \log_a(x+2)^4(x-5)^3(2x+1)^2$$

Solving logarithmic equations

Example 6

Solve for x : $\log_3(7x+3) = \log_3(5x+9)$

Solution

Using the rule $\log_b M = \log_b N$ then $M = N$

$$\text{Given } \log_3(7x+3) = \log_3(5x+9)$$

$$\text{Then } 7x+3 = 5x+9$$

$$\text{Therefore, } x = 3$$

Example 7

Solve for x : $\log_7(x-2) + \log_7(x+3) = \log_7 14$

Solution

$$\text{Given } \log_7(x-2) + \log_7(x+3) = \log_7 14$$

$$\log_7(x-2)(x+3) = \log_7 14$$

$$(x-2)(x+3) = 14$$

$$x^2 + x - 20 = 0$$

$$\text{Therefore } x = 4 \text{ or } x = -5$$

Example 8

Simplify $\log_4 27^{3/4} + \log_4 9^{3/2} - \log_4 81$

Solution

$$\text{Given } \log_4 27^{3/4} + \log_4 9^{3/2} - \log_4 81$$

$$\text{Using laws } \log_4 \left(\frac{27^{\frac{3}{4}} \times 9^{\frac{3}{2}}}{81} \right) \text{ thus } \log_4 \left(\frac{3^{\frac{9}{4}} \times 3^3}{3^4} \right)$$

$$\text{Therefore } \log_4 \left(3^{\frac{5}{4}} \right) = \frac{5}{4} \log_4 3$$

Example 9

Solve for x : $\ln(6x-5) = 3$

Solution

Given $\ln(6x-5)=3$

But $\ln x = \log_e x$

$$\log_e(6x-5)=3$$

$$x = \frac{1}{6}(e^3 + 5)$$

EXERCISE 1: ALGEBRA

Solve for x from the following

1. $7^{3x-5} = 7^{4x+3}$

2. $\left(\frac{4}{9}\right)^{3x+1} = \left(\frac{81}{16}\right)^x$

3. $25^{2x-1} = 125^{3x+4}$

4. $e^{4x-9} = 56$

5. $7^{5x+3} = 512$

6. $8^{2x-3} = \left(\frac{1}{16}\right)^{x-2}$

7. $4^{3x+5} = 8^{4x-3}$

8. $9^{2x+3} = 81^{-3x-1}$

9. $3^{2x-5} = 87$

10. $4e^{3x-5} = 195$

11. $\log_2(5x+7) = 5$

12. $\ln(4x+1) = 2$

13. $\log_3(9x+2) = 4$

14. $\log_5 x + \log_5(x-12) = 3$

15. $\log_6(x+4) + \log_6(x-2) = \log_6 4x$

17. $\log_2(x+5) - \log_2(2x-1) = 5$

16. $\log_5(4x+11) = 2$

18. $\log_8 x + \log_8(x+6) = \log_8(5x+12)$

19. $2\log 3 + \log x = \log 36$

20. $\left(\frac{1}{6}\right)^{3x+2} \times \left(\frac{1}{216}\right)^{2x} = 216^2$

21. $4^{3x+5} = 15^{x+1}$

22. $8^{2x+3} = \left(\frac{1}{4}\right)^{3x-6}$

23. $3^{2x} - 3^{x+1} + 2 = 0$

24. $\log_x 3 + \log_x 27 = 2$

25. $\log_3 x + \log_x 3 = 4$

26. Given that $\log y + 3\log x = 3$ find y

27. Simplify $\frac{27^{n+2} - 6 \times 3^{3n+3}}{3^n \times 9^{n+2}}$

28. Simplify $\frac{x}{y^{1/2} + x^{1/2}} + \frac{x}{y^{1/2} - x^{1/2}}$

29. Solve for x : $(\log_x 3)^2 + 3\log_x 3 = 4$

30. Solve for x and y : $\begin{cases} \log(x+y)=1 \\ \log_2 x + 2\log_4 y = 4 \end{cases}$
31. Solve for y : $4\log_x \sqrt{y} - \log_x 27y = \log_x y^{-2}$
32. Solve for y : $2\ln y - 3\ln x^2 = \ln \sqrt{x} + \ln x$
33. Solve for x : $\log_x 5 + 2\log_5 x = 3$
34. Solve $\log_6 x + \log_6 (x-5) = 2$
35. Simplify $\left(\frac{\left(x^2 y^{1/2} \right) (\sqrt{x}) \times \sqrt[3]{y^2}}{\left(x^5 y^3 \right)^{1/2}} \right)^{-2}$
36. Simplify $\frac{\log 25 - \log 125 + \frac{1}{2} \log 625}{3 \log 5}$
37. Simplify $\frac{x^{n-1} + x^{n+1}}{x^n}$
38. Evaluate $4^{\log_2 (2^{\log_2 5})}$
39. Evaluate $\ln \left(\frac{x^{1/2} y^2}{(xy^3)^{1/2}} \right)$

6.3 Series

If the terms of the sequence are connected by any mathematical operation such as $2 + 4 + 6 + 8 + 10 + 12 + \dots$ or $2 - 4 - 6 - 8 - \dots$ the sequence become a series.

Finite series

Finite series: Is a series with countable number of terms.

Example: $S_1: 2 + 4 + 6 + \dots + 28$, Series S_1 is finite series.

Infinite series

Infinite series: Is a series with uncountable number of terms

Example: $S_2: 2 + 4 + 6 + \dots$ this series has no end, therefore it is called infinite series.

Sigma notation, Σ

Sigma sign is a Greek letter Σ which means “summation of”. Is use to abbreviate the way of writing series.

To use the sigma sign write the n^{th} term formula or last term formula in front of the sigma notation.

The sign Σ for summation is due to Leonhard Euler (1707 – 1783) who said, “summam indicabimus signo Σ .” This symbol was used by Lagrange, but otherwise received little attention during the 8th century.

Example 10

Expand the following finite series

$$(a) \sum_{n=1}^5 n(n+1) \quad (b) \sum_{n=2}^7 \frac{1}{n^n} \quad (c) \sum_{n=1}^4 (-1)^n (n+1)$$

Solution

(a) $\sum_{n=1}^5 n(n+1)$ to expand, fix the values of n from $n=1$ to $n=5$. Remember the term $n(n+1)$ is the general term.

$$a_n = n(n+1) \text{ for } 1 \leq n \leq 5$$

$$\sum_{n=1}^5 n(n+1) = 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6$$

$$(b) \sum_{n=2}^7 \frac{1}{n^n} = \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \frac{1}{6^6} + \frac{1}{7^7}$$

$$(c) \sum_{n=1}^4 (-1)^n (n+1) = -2 + 3 - 4 + 5$$

Example 11

Expand the following infinite series up to 6th term

$$(a) \sum_{n=3}^{\infty} \left(\frac{n-2}{n} \right) \quad (b) \sum_{n=0}^{\infty} 3^{n+1} \quad (c) \sum_{n=0}^{\infty} (-1)^{n+1} n^2$$

Solution

$$(a) \sum_3^{\infty} \left(\frac{n-2}{n} \right) = \frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} \dots$$

$$(b) \sum_0^{\infty} 3^{n+1} = 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 \dots$$

$$(c) \sum_0^{\infty} (-1)^{n+1} n^2 = 0 + 1^2 - 2^2 + 3^2 - 4^2 + 5^2 \dots$$

Example 12

Re-write the following series using sigma notation

$$(a) 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3$$

$$(b) (2 \times 4) + (4 \times 6) + (6 \times 8) + \dots + (12 \times 14)$$

$$(c) \frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8} + \frac{9}{10} + \frac{11}{12} + \frac{13}{14}$$

Solution

$$(a) 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3$$

The general term of the series.

$$\text{Let } 3 + 5 + 7 + 9 + 11 + 13 + 15$$

The series is arithmetic progression with common difference $d = 2$

$$A_n = A_1 + d(n-1)$$

$$A_n = 2n + 1$$

$$\sum_{n=i}^{n=r} A_n = \sum_1^7 (2n+1)^3$$

$$(b) (2 \times 4) + (4 \times 6) + (6 \times 8) + \dots + (12 \times 14)$$

The two series here are:-

$$2 + 4 + 6 + \dots + 12 \text{ (Formed by the first numbers)}$$

$$4 + 6 + 8 + \dots + 14 \text{ (Formed by the second numbers)}$$

$$A_{n1} = 2 + 2(n-1) = 2n$$

$$A_{n2} = 4 + 2(n-1) = 2n + 2$$

$$\text{The general term is given by } \sum_{n=i}^{n=r} (A_{n1} \times A_{n2}) = \sum_{n=1}^{n=6} 2n(2n+2)$$

$$(c) \frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8} + \frac{9}{10} + \frac{11}{12} + \frac{13}{14}$$

Numerator: $1 + 3 + 5 + \dots + 13$

$$A_{n1} = 1 + 2(n-1)$$

$$A_{n1} = 2n - 1$$

$$2 + 4 + 6 + \dots + 14$$

Denominator: $2 + 4 + 6 + \dots + 14$

$$A_{n2} = 2 + 2(n-1)$$

$$A_{n2} = 2n$$

The general term is
$$\sum_{n=i}^{n=r} \left(\frac{A_{n1}}{A_{n2}} \right) = \sum_{n=1}^{n=7} \left(\frac{2n-1}{2n} \right)$$

6.4 Arithmetic and geometric progression

6.4.1 Arithmetic progression

Arithmetic progression is the series in which the difference between two consecutive terms is the same non-zero number. The non-zero difference is *common difference* and it is denoted by d .

Let A_1 be the first term of the series, the second term is obtained by adding the common difference to the first term, the third term is obtained by adding the common difference to the second term and so on.

Consider, series $3 + 7 + 11 + 15 + \dots$

$A_1 = 3$ and the common difference is $d = 7 - 3 = 11 - 7 = 15 - 11 = 4$

$$A_2 = A_1 + d = 3 + 4 = 7$$

$$A_3 = A_2 + d = 7 + 4 = 11$$

$$A_4 = A_3 + d = 11 + 4 = 15$$

$$A_5 = A_4 + d = 15 + 4 = 19 \text{ and so on.}$$

General term of arithmetic progression

Let us consider the series, $2 + 5 + 8 + \dots$ which is an arithmetic progression with common difference, $d = 3$

If $A_1 = A_1$

When $n = 2$: $A_2 = A_1 + d$

When $n = 3$: $A_3 = A_2 + d = A_1 + 2d$

When $n = 4$: $A_4 = A_3 + d = A_1 + 3d$

When $n = 5$: $A_5 = A_4 + d = A_1 + 4d$

When $n = n$: $A_n = A_{n-1} + d = A_1 + (n-1)d$

Therefore, the general term of arithmetic progression is $A_n = A_1 + (n-1)d$

Example 13

If the 10th term and the 31st term of an arithmetic progression are -15 and -57 respectively, find the

- (a) The first term and the common difference
- (b) The 101st term.

Solution

- (a) Given $A_{10} = -15$ and $A_{31} = -57$

$$A_{10} = A_1 + 9d = -15$$

$$A_{31} = A_1 + 30d = -57$$

Solve simultaneous equation is $\begin{cases} A_1 + 9d = -15 \\ A_1 + 30d = -57 \end{cases}$

The first term is $A_1 = 3$ and the common difference is $d = -2$

- (b) The 101st term, from $A_n = A_1 + (n-1)d$ thus $A_{101} = 3 + (101-1)$
therefore $A_{101} = -197$

Example 14

Determine the number of terms of an arithmetic series $5 + 11 + 17 + \dots + 119$

Solution

Given $5 + 11 + 17 + \dots + 119$

The first term, $A_1 = 5$

The last term, $A_n = 119$

The common difference, $d = 11 - 5 = 6$

From $A_n = A_1 + (n-1)d$
 $119 = 5 + (n-1)6$
 $n = 20$

Example 15

If $x + y + z = 1$ and $\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y}$ are the terms in arithmetic

progression, prove that $\frac{1}{y+z} + \frac{1}{x+z} + \frac{1}{x+y}$ are also in arithmetic progression.

Solution

Given $\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y}$

Since the terms are in A.P

$$\frac{y}{x+z} - \frac{x}{y+z} = \frac{z}{x+y} - \frac{y}{x+z} \text{ thus}$$

$$\left(\frac{y}{x+z} + 1\right) - \left(\frac{x}{y+z} + 1\right) = \left(\frac{z}{x+y} + 1\right) - \left(\frac{y}{x+z} + 1\right)$$

$$\left(\frac{x+y+z}{x+z}\right) - \left(\frac{x+y+z}{y+z}\right) = \left(\frac{x+y+z}{x+y}\right) - \left(\frac{x+y+z}{x+z}\right)$$

But $x+y+z=1$

$$\frac{1}{x+z} - \frac{1}{y+z} = \frac{1}{x+y} - \frac{1}{x+z}$$

Hence $\frac{1}{y+z} + \frac{1}{x+z} + \frac{1}{x+y}$ are in arithmetic progression

Example 16

The m times the m th term of an arithmetic progression is equal to n times the n th term of the same progression, find the $(m+n)$ th term.

Solution

Given $mA_m = nA_n$

$$m(A_1 + (m-1)d) = n(A_1 + (n-1)d)$$

$$mA_1 + (m^2 - m)d = nA_1 + (n^2 - n)d$$

$$(m-n)A_1 = (n^2 - n - m^2 + m)d$$

$$(m-n)A_1 = (n^2 - m^2 + m - n)d$$

$$(m-n)A_1 = (-(m-n)(m+n) + (m-n))d$$

$$(m-n)A_1 = -(m-n)((m+n)-1)d$$

$$A_1 + (m+n-1)d = 0 \text{ thus } A_{m+n} = A_1 + (m+n-1)d = 0$$

Therefore the $A_{m+n} = 0$

EXERCISE 2: ALGEBRA

- Find the n^{th} term of each series below
 - $1 - \frac{7}{3} - \frac{17}{3} - \dots$
 - $\frac{2}{3} + \frac{23}{12} + \frac{19}{6} + \dots$
 - $-x - 3x - 5x - \dots$
- Evaluate the n^{th} term of question 1 above when $n = 21$
- How many terms are in the following series
 - $3 + 4.5 + 6 + \dots + 49.5$
 - $107.2 + 100 + 92.8 + \dots - 605.6$
 - $16 + 19 + 22 + \dots + 79$
- The n^{th} term of an arithmetic progression is $A_n = an + b$, show that the common difference, $d = a$
- Find the median term(s) of the following arithmetic series
 - $2 + 6 + 10 + \dots + 74$
 - $-1.5 - 2.5 - 3.5 - \dots - 28.5$
- Write down the 31^{st} term of the series
 - $\frac{2}{3} + \frac{7}{6} + \frac{5}{3} + \dots$
 - $13 + 7 + 1 + \dots$
- The n^{th} of an arithmetic progression is given as $a_n = 2n - 3$ evaluate the 100^{th} term
- The 4^{th} and 28^{th} terms of an arithmetic progression are 6 and 78 respectively; find the first term and the common difference.
- Find the n^{th} of the progression: $\log 3 + \log 6 + \log 12 + \dots$
- If $\ln\left(\frac{x^2}{2}\right) + \ln\left(\frac{x^3}{4}\right) + \ln\left(\frac{x^4}{8}\right) + \dots$ are in arithmetic progression, show that the n^{th} is $(n+1)\ln x - n\ln 2$

The sum of the first n^{th} terms of an arithmetic progression

Consider the following arithmetic series: $A_1 + A_2 + A_3 + \dots + A_n$, the sum of n^{th} terms is denoted as S_n , therefore, $S_n = A_1 + A_2 + A_3 + \dots + A_n$

$$+ \begin{cases} S_n = A_1 + A_1 + d + A_1 + 2d + \dots + A_n \\ S_n = A_n + A_n - d + A_n - 2d + \dots + A_1 \end{cases}$$

$$2S_n = (A_1 + A_n) + (A_1 + A_n) + (A_1 + A_n) + \dots + (A_1 + A_n)$$

$$S_n = \frac{n}{2}(A_1 + A_n) \text{ But } A_n = A_1 + d(n-1)$$

$$\text{The sum is } S_n = \frac{n}{2}(2A_1 + d(n-1))$$

Example 17

The 24th term of an A.P is 54; find the sum of the first 47 terms.

Solution

$$\text{Given } A_{24} = A_1 + 23d = 54 \text{ and } S_{47} = \frac{47}{2}(2A_1 + 46d)$$

$$\text{Simplifying } S_{47} = 47(A_1 + 23d)$$

$$\text{Therefore, } S_{47} = 47 \times 54 \Rightarrow S_{47} = 2538$$

Sum of 47 terms is 2538.

Example 18

The first term of an arithmetic progression is 10 the last term is 70. The sum of all the terms is 800 find the common difference.

Solution

$$\text{Given } A_1 = 10, A_n = 70 \text{ and } S_n = 800$$

$$\text{Using } S_n = \frac{n}{2}(A_1 + A_n)$$

$$800 = \frac{n}{2}(10 + 70) \Rightarrow n = 20$$

$$A_n = A_1 + (n-1)d$$

$$70 = 10 + 19d$$

$$60 = 19d \Rightarrow d = \frac{60}{19}, \text{ the common difference is } \frac{60}{19}$$

Example 19

The sum of n^{th} terms of an arithmetic progression is x , the n^{th} term is y , Show that

(a) The first term is $A_1 = \left(\frac{2x - ny}{n} \right)$

(b) The common difference is $d = \frac{2}{n} \left(\frac{y - x}{n - 1} \right)$

Solution

(a) Given $S_n = x$, $A_n = y$ and $S_n = \frac{n}{2}(A_1 + A_n)$

Substituting, $x = \frac{n}{2}(A_1 + y)$

$$2x = nA_1 + ny \Rightarrow A_1 = \frac{2x - ny}{n} \text{ shown.}$$

(b) $S_n = \frac{n}{2}(2A_1 + d(n - 1))$

$$x = \frac{n}{2} \left(2 \left(\frac{2x - ny}{n} \right) + d(n - 1) \right)$$

$$2x = n \left(\frac{4x - 2ny}{n} + d(n - 1) \right)$$

$$2x = 4x - 2ny + d(n^2 - n)$$

$$d = \frac{2ny - 2x}{n^2 - n} = \frac{2}{n} \left(\frac{y - x}{n - 1} \right)$$

Example 20

The sum of $3n$ terms of the series $20 + 23 + 29 + \dots$ is equal to the sum of the series $a + (a + d) + (a + 2d) + (a + 3d) + \dots$ for all values of n . Show that $d = \frac{27}{4}$ and find the value of a .

Solution

$$S_{3n} : 20 + 23 + 26 + 29 + \dots = \frac{3n}{2} (2 \times 20 + 3(3n - 1))$$

$$S_{2n} : a + (a + d) + (a + 2d) + (a + 3d) + \dots = \frac{2n}{2} (2a + d(2n - 1))$$

Given $S_{3n} = S_{2n}$

$$\frac{3n}{2}(37+9n) = n(2a + (2n-1)d)$$

The identity holds for all values of n .

When $n=1$

$$\frac{3}{2}(37+9) = 2a + d$$

$$2a + d = 69 \quad (i)$$

When $n=2$

$$3(37+18) = 2(2a + 3d)$$

$$4a + 6d = 165 \quad (ii)$$

$$\begin{cases} 2a + d = 69 \\ 4a + 6d = 165 \end{cases}$$

Solve simultaneously equations (i) and (ii)

Therefore, $d = \frac{27}{4}$ and $a = \frac{249}{8}$ Hence shown.

Example 21

The 6th and 31st terms of an arithmetic progression are -26 and -101 respectively. Find

- The first term and the common difference
- If there are 100 terms in this series, find the sum of the last 20 terms

Solution

- (a) Given $A_6 = -26$ and $A_{31} = -101$

Writing A_6 and A_{31} in terms of A_1 and d , solve simultaneously

$$\begin{cases} A_1 + 5d = -26 \\ A_1 + 30d = -101 \end{cases}$$

$$25d = -75 \Rightarrow d = -3$$

$$A_1 = -26 + 15 = -11$$

Therefore, the first term is -11 and the common difference is -3.

- (b) The sum of the last 20 terms

$$A_{81} = A_1 + 80d \Rightarrow A_{81} = -11 + 80(-3) = -251$$

$$A_{100} = A_1 + 99d \Rightarrow A_{100} = -11 + 99(-3) = -308$$

$$S_n = \frac{n}{2}(A_{81} + A_{100})$$

$$S_{20} = \frac{20}{2}(-251 + -308)$$

$$S_{20} = -5590$$

The sum of the last 20 terms is -5590

Example 22

Find the sum of the first 100 terms of the series $\log 3 + \log 6 + \log 12 + \dots$

Solution

Given $\log 3 + \log 6 + \log 12 + \dots$

$$d = \log 6 - \log 3 = \log 12 - \log 6$$

$$d = \log\left(\frac{6}{3}\right) = \log\left(\frac{12}{6}\right) = \log 2$$

$$S_n = \frac{n}{2}(2A_1 + d(n-1))$$

$$S_{100} = \frac{100}{2}(2\log 3 + \log 2(100-1))$$

$$S_{100} = 50(2\log 3 + 99\log 2)$$

$$S_{100} = 50\log(9 \times 2^{99})$$

Example 23

The terms $\log 5 + \log x + \log y + \log z + \log 80$ are in arithmetic progression, find the values of x , y and z .

Solution

$$\log 5 + \log x + \log y + \log z + \log 80$$

$$\text{The common difference, } d = \log x - \log 5 = \log\left(\frac{x}{5}\right)$$

$$\text{The 5}^{\text{th}} \text{ term is given, } A_5 = A_1 + 4d \Rightarrow \log 5 + 4\log\left(\frac{x}{5}\right) = \log 80$$

$$4\log\left(\frac{x}{5}\right) = \log\left(\frac{80}{5}\right)$$

$$4\log\left(\frac{x}{5}\right) = \log 16$$

$$4\log x - 4\log 5 = \log 16$$

$$4\log x = \log 16 + \log 5^4$$

$$4\log x = \log(16 \times 625)$$

$$4\log x = \log 10000$$

$$4\log x = 4$$

$$\log x = 1 \Rightarrow x = 10^1 = 10$$

$$A_2 = \log y = A_1 + d$$

$$\log y = \log 5 + \log\left(\frac{x}{5}\right)$$

But

$$x = 10$$

$$\log y = \log 5 + \log 2 \Rightarrow \log y = \log(5 \times 2)$$

$$\log y = \log 10 \Rightarrow y = 10$$

EXERCISE 3: ALGEBRA

- The lengths of the sides of a right-angled triangle make an arithmetic progression. The longer side is 24 cm long. Find out the perimeter of the triangle.
- An arithmetic progression consists of 8 terms. Sum of the middle terms is 41; the 8th term minus the first term is 112. Find the third term.
- Iron pipes are stored in 20 rows. The top row contains of 130 pipes, next row always contains of two more pipe than the previous one. How many pipes are there?
- The angles in a triangle make an arithmetic progression. The smallest angle is 20°. Determine sizes of the other angles.
- The angles of an irregular pentagon are in arithmetic progression. The largest angle is 148°. Determine all angles.
- Length of the sides of a cuboid make 3 terms of an arithmetic progression. Sum of the length equals 24 cm; the volume of the cuboid equals 312 cm³. Determine the length of the sides.
- How long a rock be falling to a mine somewhere in East Africa, which is deep 2500 m if it moves 4,904 m in the first minute and extra 9,808 m for every next minute?
- If $\log_2 x + \log_2(x+9) + \log_2(x+45) + \dots$ are in arithmetic progression, find
 - The value of x.
 - The first term and the common difference; and
 - The 5th term
 - The sum of the first 10 terms as a single logarithm.

-
9. Find the sum of all natural numbers between 98 to 1000 which are multiple of 3
 10. The ratio of the sums of m and n terms of an arithmetic progression is $m^2 : n^2$. Show that the ratio of m^{th} and n^{th} terms is $(2m-1):(2n-1)$
 11. The sum of n terms of a progression is $S_n = 2n^2 + 5n$. Show that the progression is arithmetic with the first term 7.
 12. In an arithmetic progression, the n^{th} term is $\frac{1}{m}$ and the m^{th} term is $\frac{1}{n}$, find
 - (a) The $(mn)^{\text{th}}$ term
 - (b) The sum of the first mn terms.
 13. The sum of n , $2n$ and $3n$ terms of an arithmetic progression are S_1 , S_2 and S_3 . Show that $S_3 = 3(S_2 - S_1)$
 14. Find the sum of all-natural numbers between 1 and 200, which are divisible by three.
 15. The sum of three consecutive terms of an arithmetic progression is 27; the product of the same terms is 504. Find the terms
 16. Insert 4 terms between the roots of the quadratic equation $x^2 - 16x + 39 = 0$ so that they will make an arithmetic progression.
 17. A man takes an interest free loan of Tshs. 1740 from his friend agreeing to repay in monthly instalment. He gives Tshs. 200 in the first month and diminishes his monthly instalments by Tshs. 10 each month. How many months will it take to repay the loan?
 18. How many terms of the progression $3 + 6 + 9 + \dots$ must be taken to have a sum not less than 2000?
 19. In staff part at Ngurdoto mountain lodge 2017, eggs placed 1 meter apart in a straight line. A competitor starts from a point in the line, which is 5 metre from the nearest egg. Find an expression for the total distance run in collecting the eggs, one at a time and bringing them back one at a time to the starting point. Calculate the value n if the total distance is 162 metres.
 20. If the sum of the first n terms of a series is $S_n = an^2 + bn$, show that the series is an arithmetic progression and find the common difference.
 21. An arithmetic progression is such that the 7th term is three times the sum of the third and fourth terms. If the sum of the first 6 terms is 21, find
 - (a) Find the first term and the common difference
 - (b) The sum of the first 50 terms
-

Arithmetic Mean (A.M)

Let a , M and b be three consecutive terms of an arithmetic mean $a + M + b$ are in A.P. The common difference, $d = M - a = b - M$

$$2M = a + b \Rightarrow M = \frac{a+b}{2} \leftarrow \text{is an arithmetic mean.}$$

Example 24

Find the arithmetic mean between 13.5 and 19.

Solution

$$A.M = \frac{a+b}{2} = \frac{13.5+19}{2} = 16.25$$

The arithmetic mean is 16.25.

Example 25

Insert five arithmetic mean between 8 and 26

Solution

We need to find the 5 missing terms of the arithmetic progression

There are seven terms here, where $A_1 = 20$ and $A_7 = 26$

$$26 = 8 + (7-1)d \text{ thus } 26 - 8 = 6d \Rightarrow d = 3$$

$$A_2 = 8 + 3 = 10, A_3 = 8 + 6 = 14, A_4 = 8 + 9 = 17, A_5 = 8 + 12 = 20,$$

$$A_6 = 8 + 15 = 23$$

The terms are 10, 14, 17, 20 and 23.

Example 26

The n arithmetic means between 20 and 80 are such that the ratio of the first mean and the last mean is 1:3. Find the value of n .

Solution

The first term is 20 and the last term is 80, but the 80 is $(n+2)^{th}$ term.

$$A_{n+2} = A_1 + (n+2-1)d$$

$$20 + (n+1)d = 80 \Rightarrow d = \frac{60}{n+1}$$

The first arithmetic mean (the second term)

$$A.M_F = A_2 = 20 + \frac{60}{n+1} = \frac{20n+80}{n+1}$$

The last arithmetic mean, (the $(n+1)^{th}$ term)

$$A_{n+1} = A.M = A_1 + (n-1)d$$

$$A.M_L = 20 + n\left(\frac{60}{n+1}\right) = \frac{80n+20}{n+1}$$

The ratio between the first arithmetic mean and the last arithmetic mean is 1:3

(The ratio between the second term and the $(n+1)^{th}$ term is 1:3)

$$A.M_F : A.M_L = 1:3 \text{ or } A_2 : A_{n+1} = 1:3$$

$$\frac{20n+80}{n+1} : \frac{80n+20}{n+1} = 1:3$$

$$\frac{20n+80}{n+1} \times \frac{n+1}{80n+20} = \frac{1}{3}$$

$$\frac{n+4}{4n+1} = \frac{1}{3} \Rightarrow 3n+12 = 4n+1$$

The value of n is 11.

EXERCISE 4: ALGEBRA

- Find the arithmetic mean between 43 and 47
- Insert 8 arithmetic mean between 1 and 37
- The arithmetic mean between 18 and $2k$ is 21, find the value of k .
- Prove that if the number of terms of an arithmetic progression is odd then the middle term is the arithmetic mean of the first and the last term.
- Between 7 and 85, m numbers of arithmetic means are inserted so that the ratio of the $(m-3)^{th}$ and m^{th} means is 11:24. Find the value of m .
- Prove that the sum of n arithmetic means between two numbers is n times the single arithmetic mean between them.
- The arithmetic mean between n^{th} and m^{th} terms of an arithmetic progression is equal to the arithmetic mean between p^{th} and q^{th} terms of an arithmetic progression, show that $n+m=p+q$
- There are n arithmetic mean between 3 and 17. The ratio between the last to the first mean is 3:1. Find the value of n .
- Find two numbers whose product is 96 and whose arithmetic mean is 10
- Find the arithmetic mean between $(a+b)^2$ and $(a-b)^2$, hence evaluate the value of the expression when $a = \cos \alpha$ and $b = \sin \alpha$

6.4.2 Geometric progression (G.P)

Geometric progression is a progression in which the ratio of the second term to the first term is equal to the ratio of the fourth term to the third term and so on; the ratio is called common ratio denoted r .

Consider the series: $2 + 4 + 8 + 16 + 32 + \dots$

$$\text{Common ratio, } r = \frac{G_2}{G_1} = \frac{G_3}{G_2} = \frac{4}{2} = \frac{8}{4} = 2$$

The first term, $G_1 = 2$

The second term $G_2 = G_1 r = 2 \times 2 = 4$

The third term $G_3 = G_1 r^2 = 2 \times 2^2 = 8$

The fourth term $G_4 = G_1 r^3 = 2 \times 2^3 = 16$

The fifth term $G_5 = G_1 r^4 = 2 \times 2^4 = 32$

Then the n^{th} term is $G_n = G_1 r^{n-1}$

The n^{th} term formula of geometric progression is $G_n = G_1 r^{n-1}$

Example 27

Find the 7^{th} and 11^{th} terms of the geometric progression $3 + 12 + 48 + \dots$

Solution

Given $3 + 12 + 48 + \dots$

$$\text{The common ratio, } r = \frac{12}{3} = \frac{48}{12} = 4$$

The 7^{th} term, $G_7 = G_1 r^6 = 3 \times 4^6 = 768$

The 11^{th} term, $G_{11} = G_1 r^{10} = 3 \times 4^{10}$

Example 28

The 4^{th} and 9^{th} terms of a geometric progression are 8 and 256 respectively.

Find the first 5 terms.

Solution

$$\begin{cases} G_4 = 8 \\ G_9 = 256 \end{cases} \text{ then } \begin{cases} G_1 r^3 = 8 \\ G_1 r^8 = 256 \end{cases}$$

$$G_1 = \frac{8}{r^3} \text{ Substituting in second equation } \left(\frac{8}{r^3} \right) r^8 = 256$$

$$8r^5 = 256$$

Therefore, $r^5 = 32 \Rightarrow r^5 = 2^5 \Rightarrow r = 2$

The first term, $G_1 = \frac{8}{2^3} = 1$

The first five terms are $G_2 = 2$, $G_3 = 4$, $G_4 = 8$, $G_5 = 16$

The sum of the first n^{th} terms of geometric progression

The sum of the first n terms of a geometric is denoted as S_n , consider the sum of the terms below

$$S_n = G_1 + G_2 + G_3 + \dots + G_n$$

$$S_n = G_1 + G_1r + G_1r^2 + \dots + G_1r^{n-2} + G_1r^{n-1} \quad (1)$$

Multiplying (1) by r

$$rS_n = G_1r + G_1r^2 + G_1r^3 + \dots + G_1r^{n-2} + G_1r^{n-1} + G_1r^n \quad (2)$$

Take (2) minus (1)

$$\begin{aligned} - \left\{ \begin{aligned} rS_n &= G_1r + G_1r^2 + G_1r^3 + \dots + G_1r^{n-2} + G_1r^{n-1} + G_1r^n \\ S_n &= G_1 + G_1r + G_1r^2 + G_1r^3 + \dots + G_1r^{n-2} + G_1r^{n-1} \end{aligned} \right. \\ rS_n - S_n &= G_1r^n - G_1 \end{aligned}$$

Make S_n the subject $(r-1)S_n = G_1(r^n - 1)$

Hence the sum is $S_n = G_1 \left(\frac{r^n - 1}{r - 1} \right)$, for all $|r| > 1$

If $|r| < 1$, the sum is given by $S_n = G_1 \left(\frac{1 - r^n}{1 - r} \right)$

If $|r| < 1$ and $\lim_{n \rightarrow \infty} (r^n) \rightarrow 0$ the formula deduced to $S_\infty = \frac{G_1}{1 - r}$ this is sum to

infinity formula. This formula may also be used to convert recurring decimals to fractions as a sum to infinity series.

Example 29

The third term and seventh term of a geometric progression is 45 and 3645 respectively, if there are 30 terms

- (a) The sum of the first 10 terms
 (b) The sum of the last ten terms

Solution

$$(a) \quad G_3 = G_1 r^2 = 45 \quad (1)$$

$$G_7 = G_1 r^6 = 3645 \quad (2)$$

Use (1) to make G_1 the subject, $G_1 = \frac{45}{r^2}$

$$\text{Substitute in (2) above } \left(\frac{45}{r^2}\right) r^6 = 3645$$

$$\text{The common ratio is } r^4 = 81 = 3^4 \Rightarrow r = 3$$

$$\text{The first term, } G_1 = \frac{45}{3^2} = 5$$

$$\text{Sum of the } n \text{ terms, } S_n = G_1 \left(\frac{r^n - 1}{r - 1} \right) \text{ for } |r| > 1$$

$$\text{The sum of the first 10 terms, } S_{10} = 5 \left(\frac{3^{10} - 1}{3 - 1} \right) = 147620$$

- (b) The sum of the last term terms, The first term become the eleventh term

$$S_{10} = G_{11} \left(\frac{r^{10} - 1}{r - 1} \right) \text{ thus } S_{10} = 5 \times 3^{10} \left(\frac{3^{10} - 1}{3 - 1} \right)$$

$$\text{The sum of the last 10 terms is } S_{10} = \frac{5}{2} (3^{20} - 3^{10})$$

Example 30

Find the sum of the first 20 terms of the series $81 + 27 + 9 + \dots$ and hence show that as $n \rightarrow \infty$ then $S_n \rightarrow S_\infty$

Solution

$$\text{The common ratio, } r = \frac{G_2}{G_1} = \frac{27}{81} = \frac{1}{3}$$

$$S_n = 81 \left(\frac{1 - \left(\frac{1}{3}\right)^{20}}{1 - \frac{1}{3}} \right) = 121.5$$

$$S_\infty = \frac{G_1}{1-r}$$

$$S_\infty = \frac{81}{\left(1 - \frac{1}{3}\right)} = 121.5$$

Hence $S_n \rightarrow S_\infty$

Example 31

Deduce the sum of first n terms of the series $5 + 55 + 555 + \dots$

Solution

Given $5 + 55 + 555 + \dots$ factorize 5 from the terms

$5(1 + 11 + 111 + \dots)$ multiply the terms by 9 and divide by 9 to keep the series unchanged

$$\frac{5}{9}(9 + 99 + 999 + \dots)$$

$$\frac{5}{9}((10-1) + (100-1) + (1000-1) + \dots)$$

$$\frac{5}{9}(10 + 100 + 1000 + \dots) - (1 + 1 + 1 + \dots)$$

The series $10 + 100 + 1000 + \dots$ is geometric progression with $r = 10$

The series $1 + 1 + 1 + \dots$ is an arithmetic progression with the $d = 0$

The sum become, $S_n = \frac{5}{9} \left(10 \left(\frac{10^n - 1}{10 - 1} \right) - n \right)$

The sum is $S_n = \frac{50}{81} (10^n - 1 - 9n)$

Example 32

Deduce the formula for the sum of $0.3 + 0.33 + 0.333 + \dots$

Solution

Given $0.3 + 0.33 + 0.333 + \dots$ thus $3(0.1 + 0.11 + 0.111 + \dots)$

$$\frac{1}{3}(0.9 + 0.99 + 0.999 + \dots)$$

$$\frac{1}{3}((1-0.1) + (1-0.01) + (1-0.001) + \dots)$$

$$\frac{1}{3}((1+1+1+\dots)-(0.1+0.01+0.001+\dots))$$

$$S_n = \frac{1}{3} \left(n - \frac{1}{10} \left(\frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}} \right) \right)$$

$$S_n = \frac{1}{3} \left(n - \frac{1}{9} \left(\frac{10^n - 1}{10^n} \right) \right)$$

$$S_n = \frac{1}{27} \left(9n - \frac{10^n - 1}{10^n} \right)$$

Using sum to infinity formula to convert recurring decimals to fractions

Example 33

Express the recurring decimal $0.\dot{6}$ as an infinity geometric progression

Solution

Given $0.\dot{6} = 0.666\dots$

$$0.\dot{6} = 0.6 + 0.06 + 0.006 + \dots$$

Common ratio, $r = \frac{0.06}{0.6} = 0.1$

Sum to infinity, $S_\infty = \frac{G_1}{1-r}$

$$S_\infty = \frac{0.6}{1-0.1} = \frac{2}{3}$$

Therefore, $0.\dot{6} = \frac{2}{3}$

Example 34

Express $0.\dot{4}\dot{1}$ as a fraction using sum to infinity formula

Solution

Given $0.\dot{4}\dot{1} = 0.414141\dots$

$$0.\dot{4}\dot{1} = 0.41 + 0.0041 + 0.000041 + \dots$$

$$0.\dot{4}\dot{1} = \frac{41}{100} + \frac{41}{10000} + \frac{41}{1000000} + \dots$$

$$S_{\infty} = \frac{\frac{41}{100}}{1 - \frac{1}{100}} = \frac{41}{100} \times \frac{100}{99} = \frac{41}{99}$$

Therefore, $0.\dot{4}\dot{1} = \frac{41}{99}$

Example 35

Convert $2.4\dot{3}$ to fraction using sum to infinity limit

Solution

Given $2.4\dot{3} = 2.43333\dots$

$$2.4 + \left(\frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots \right)$$

The series $\frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots$ is a converging geometric series.

$$S_n = 2 + \frac{4}{10} + \left(\frac{\frac{3}{100}}{1 - \frac{1}{10}} \right) = 2 + \frac{4}{10} + \frac{1}{30} = \frac{73}{30}$$

Example 36

Express $0.31\dot{3}$ as a fraction

Solution

From $0.31\dot{3} = 0.31333\dots$

$$\frac{3}{10} + \frac{1}{100} + \left(\frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \dots \right)$$

$$S_{\infty} = \frac{G_1}{1-r} \text{ thus } S_n = \frac{3}{10} + \frac{1}{100} + \left(\frac{\frac{3}{1000}}{1 - \frac{1}{10}} \right)$$

$$S_{\infty} = \frac{47}{150}$$

Geometric Mean (G. M)

If a, G and b are three consecutive terms of geometric progression, then

The common ratio $\frac{b}{G} = \frac{G}{a} \Rightarrow G^2 = ab \Rightarrow G = \sqrt{ab}$

The geometric mean is $G.M = \sqrt{ab}$

The geometric mean of n terms is defined as the n^{th} root of their product.

$$G.M = (a_1 \times a_2 \times a_3 \times \dots \times a_n)^{\frac{1}{n}}$$

Example 37

Find the geometric mean between $\frac{3}{4}$ and $\frac{27}{4}$

Solution

Geometric mean is $G.M = \sqrt{\frac{3}{4} \times \frac{27}{4}} = \sqrt{\frac{81}{16}} = \frac{9}{4}$

The geometric mean is $\frac{9}{4}$

Example 38

Insert the missing terms in $4 + _ + 36 + _ + 324$ so that the series become a geometric progression.

Solution

The geometric mean between 4 and 36 and the geometric mean between 36 and 324.

The $G.M = \sqrt{4 \times 36} = \sqrt{144} = 12$ and the $G.M = \sqrt{36 \times 324} = 108$

The terms are 12 and 108.

Example 39

Insert 8 geometric means between 8 and $\frac{1}{64}$

Solution

The simple way is to find the 8 terms between 8 and $\frac{1}{64}$ which makes a geometric progression.

The progression has 10 terms, where the first term, $G_1 = 8$ and the last term is

$$G_{10} = \frac{1}{64}$$

Form $G_n = G_1 r^{n-1}$ thus $G_{10} = G_1 r^9$ therefore $\frac{1}{64} = 8 \times r^9$

$$r^9 = \frac{1}{512}, \quad r^9 = \frac{1}{2^9} = \left(\frac{1}{2}\right)^9 \Rightarrow r = \frac{1}{2}$$

$$G_2 = 8 \times \frac{1}{2} = 4, \quad G_3 = 8 \times \frac{1}{2^2} = 2, \quad G_4 = 8 \times \frac{1}{2^3} = 1, \quad G_5 = 8 \times \frac{1}{2^4} = \frac{1}{2},$$

$$G_6 = 8 \times \frac{1}{2^5} = \frac{1}{4}, \quad G_7 = 8 \times \frac{1}{2^6} = \frac{1}{8}, \quad G_8 = 8 \times \frac{1}{2^7} = \frac{1}{16} \text{ and}$$

$$G_9 = 8 \times \frac{1}{2^8} = \frac{1}{32}. \text{ The terms are } 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \text{ and } \frac{1}{32}$$

The relation between arithmetic and geometric mean

If a and b are two terms in a progression, where $a > b$, their arithmetic mean is always greater than their geometric mean.

We are required to show that $\frac{a+b}{2} > \sqrt{ab}$, provided $a > b$

Consider $a > b$

$$a - b > 0$$

Square both sides

$$(a - b)^2 > 0$$

$$a^2 - 2ab + b^2 > 0 \text{ then } a^2 + b^2 > 2ab$$

$$(a + b)^2 - 2ab > 2ab$$

$$(a + b)^2 > 4ab$$

$$a + b > 2\sqrt{ab}$$

Therefore, $\frac{a+b}{2} > \sqrt{ab}$

EXERCISE 5: ALGEBRA

1. In a geometric progression, the third term exceed the second by 8, the sum of the first three terms is 28. Find the sum of the first 10 terms.
2. The first three terms of geometric progression are $-5, 1, -\frac{1}{5}$ find the n^{th} term.
3. Given the series $3+y+z+w+48$ find the missing values so that the series become a geometric series.
4. The sum of three consecutive terms of a geometric progression is $\frac{7}{8}$, the product of the same terms is $\frac{1}{64}$, find the terms.
5. Find the sum to infinity $1 + \frac{1}{3} + \frac{1}{9} + \dots$
6. A ball falls a height of 3.2 m. If after each fall it re-bounce three-quarter the distance it has fallen, how high the ball re-bounce in fifth time 222.
7. Find the first term and the common ratio of geometric progression whose third term is 3 and the eighth term is -96.
8. A well driller charges Tshs. 2500 for drilling the first metre of a well and for every metre thereafter he charges Tshs. 500 more than the preceding metre. How much does the charges for drilling 100 meter well?
9. The third terms exceed the first by 25% of the third term, show that $r = \pm \frac{2\sqrt{3}}{3}$
10. The sum to infinity of a geometric series is 16, if the second term is 4. Find
 - (a) The first term
 - (b) The common ratio
11. Convert the following recurring decimals into fractions by using the infinity geometric progression approach

a) $0.\dot{2}$	b) $0.\dot{2}\dot{1}$	c) $3.\dot{2}$
d) $0.\dot{7}$	e) $0.3\dot{7}$	f) $0.\dot{3}0\dot{4}$
g) $2.9\dot{8}$	h) $7.23\dot{4}$	
12. The sum of the first 2 terms of a geometric progression is 9 and the sum of the first four terms is 45. Calculate the common ratio and the first term
13. The sum of the first n terms of progression is n^2

- (a) Write down the formula for n^{th} term
 - (b) Show that the series is an arithmetic series
 - (c) Find the sum of the first $2n$ terms
14. The second, third and seventh terms of arithmetic progression are the first three terms of a geometric progression, if the common difference is 3, find
 - (a) The common ratio.
 - (b) The sum of the first 10 terms of geometric progression
15. The third, fourth and eighth terms of an arithmetic progression forms the first three consecutive terms of Geometric progression. If the sum of the first 10 terms of A.P is 85, Calculate
 - a) First term of G.P and first term of A.P
 - b) Common ratio and common difference
 - c) The sum of the first 5 terms of the G.P
16. Chairs in a conference room are arranged such that the first row has 18 chairs and each of the preceding rows has two more chairs than the rows in the front of it. If there are 40 rows altogether, find the total number of chairs in this room.
17. Given three non zero numbers x, y and z which are consecutive terms of a G.P and $x + y + z = 70$, furthermore $4x, 5y$ and $4z$ are consecutive terms of an A.P. Find values of x, y and z
18. Non zero numbers A, B, C are consecutive terms of a G.P and $A, 2B$ and C are consecutive terms of A.P. Show that the common ratio of the G.P is $2 \pm \sqrt{3}$.
19. In a certain A.P the sum of terms from the 10^{th} to 20^{th} terms inclusive is 1155. If the first term is 7, find the common difference.
20. The first three terms of an arithmetic progression are a, b and a^2 respectively where $a \neq 0$. The first three terms of geometrical progression are a, a^2 and b . Find the values of a and b
21. Find the positive difference between the sum of the first ten terms of a geometrical and arithmetical progression which starts with $3+6+\dots$
22. Show that if $\log a, \log b$ and $\log c$ are consecutive terms of an arithmetic progression then a, b and c are the consecutive terms of a geometric progression.

23. The sum of n terms of a certain series is $4^n - 1$ for all values of n . Find the first four terms and the n^{th} term terms, and show that the progression is a geometrical progression.

6.4.3 Standard results

Series of powers of the natural numbers

1. Sum of the first n – natural numbers: $1 + 2 + 3 + 4 + \dots + n$

$$\begin{aligned}
 & \left| \begin{array}{l} S_n = 1 + 2 + 3 + \dots + (n-1) + n \\ S_n = n + (n-1) + (n-2) + \dots + 2 + 1 \end{array} \right. \\
 & 2S_n = (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) \\
 & 2S_n = n(n+1) \quad \text{thus } S_n = \frac{n}{2}(n+1)
 \end{aligned}$$

2. The sum of the first n – square natural numbers:

$$1^2 + 2^2 + 3^2 + \dots + n^2$$

$$\text{Consider the identity } A_n = (n+1)^3 - n^3 = 3n^2 + 3n + 1$$

$$\begin{array}{rcl}
 A_1 & = & 2^3 - 1^3 = 3 \cdot 1^2 + 3 \cdot 1 + 1 \\
 A_2 & = & 3^3 - 2^3 = 3 \cdot 2^2 + 3 \cdot 2 + 1 \\
 A_3 & = & 4^3 - 3^3 = 3 \cdot 3^2 + 3 \cdot 3 + 1 \\
 & \text{-----} & \\
 + & \text{-----} & \\
 & \text{-----} & \\
 A_{n-2} & = & (n-2)^3 - (n-2)^3 = 3(n-2)^2 + 3(n-2) + 1 \\
 A_{n-1} & = & (n-1)^3 - (n-2)^3 = 3(n-1)^2 + 3(n-1) + 1 \\
 A_n & = & n^3 - (n-1)^3 = 3n^2 + 3n + 1 \\
 & (n+1)^3 - 1 = \sum (3n^2 + 3n + 1) \\
 & n^3 + 3n^2 + 3n + 1 - 1 = \sum (3n^2) + \sum (3n) + \sum (1) \\
 & n^3 + 3n^2 + 3n = 3 \sum n^2 + 3 \sum n + n \\
 & n^3 + 3n^2 + 3n = 3 \sum n^2 + 3 \left(\frac{n}{2}(n+1) \right) + n
 \end{array}$$

$$n^3 + 3n^2 + 3n = 3 \sum n^2 + \frac{3n^2}{2} + \frac{3n}{2} + n$$

$$2n^3 + 6n^2 + 6n = 6 \sum n^2 + 3n^2 + 3n + 2n$$

$$2n^3 + 3n^2 + n = 6 \sum n^2$$

$$6 \sum n^2 = n(2n^2 + 3n + 1)$$

$$\sum n^2 = \frac{n}{6}(n+1)(2n+1)$$

$$S_{n^2} = \frac{n}{6}(n+1)(2n+1)$$

The sum of the square of the first natural numbers is

$$\sum n^2 = \frac{n}{6}(n+1)(2n+1)$$

3. The sum of the first n – cubic natural numbers: $1^3 + 2^3 + 3^3 + \dots + n^3$

Consider the identity $(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$

$$\left[\begin{array}{l} A_1 = 2^4 - 1^4 = 4 \times 1^3 + 6 \times 1^2 + 4 \times 1^1 + 1 \\ A_2 = 3^4 - 2^4 = 4 \times 2^3 + 6 \times 2^2 + 4 \times 2^1 + 1 \\ A_3 = 4^4 - 3^4 = 4 \times 3^3 + 6 \times 3^2 + 4 \times 3^1 + 1 \\ \text{-----} \end{array} \right.$$

$$+ \left[\text{-----} \right.$$

$$A_{n-2} = (n-1)^4 - (n-2)^4 = 4(n-2)^3 + 6(n-2)^2 + 4(n-2)^1 + 1$$

$$A_{n-1} = n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1)^1 + 1$$

$$\left[A_n = (n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1 \right.$$

$$(n+1)^4 - 1 = \sum (4n^3 + 6n^2 + 4n + 1)$$

$$n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 = 4 \sum n^3 + 6 \sum n^2 + 4 \sum n + \sum 1$$

$$n^4 + 4n^3 + 6n^2 + 4n = 4 \sum n^3 + 6 \left(\frac{n}{6} (2n^2 + 3n + 1) \right) + 4 \left(\frac{n}{2} (n+1) \right) + n$$

$$n^4 + 4n^3 + 6n^2 + 4n = 4 \sum n^3 + 2n^3 + 3n^2 + n + 2n^2 + 2n + n$$

$$n^4 + 4n^3 + 6n^2 + 4n = 4 \sum n^3 + 2n^3 + 5n^2 + 4n$$

$$\begin{aligned}
 n^4 + 2n^3 + n^2 &= 4 \sum n^3 \\
 n^2(n^2 + 2n + 1) &= 4 \sum n^3 \\
 4 \sum n^3 &= n^2(n+1)^2 \\
 \sum n^3 &= \frac{n^2}{4}(n+1)^2
 \end{aligned}$$

Therefore, $S_{n^3} = \frac{n^2}{4}(n+1)^2$

Example 40

Evaluate the following

$$(a) \sum_{r=1}^n n(n+1) \quad (b) \sum_{r=1}^n (n^2 + 3) \quad (c) \sum_{r=1}^n (n+1)(n+2)$$

Solution

(a) $\sum_{r=1}^n n(n+1)$ expanding

$$\begin{aligned}
 \sum_{r=1}^n (n^2 + n) &= \sum_{r=1}^n n^2 + \sum_{r=1}^n n \\
 &= \frac{n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1) \\
 &= n(n+1) \left(\frac{2n+1}{6} + \frac{1}{2} \right) \\
 &= \frac{n}{6}(n+1)(2n+4) \\
 \sum_{r=1}^n n(n+1) &= \frac{n}{3}(n+1)(n+2)
 \end{aligned}$$

(b) $\sum_{r=1}^n (n^2 + 3)$

$$\sum_{r=1}^n n^2 + \sum_{r=1}^n 3 = \frac{n}{6}(n+1)(2n+1) + 3n$$

$$\sum_{r=1}^n (n^2 + 3) = \frac{n}{6} (2n^2 + 3n + 19)$$

$$(c) \sum_{r=1}^n (n+1)(n+2)$$

$$\sum_{r=1}^n (n+1)(n+2) = \sum_{r=1}^n (n^2 + 3n + 2)$$

$$\sum_{r=1}^n n^2 + \sum_{r=1}^n 3n + \sum_{r=1}^n 2 = \frac{n}{6} (n+1)(2n+1) + \frac{3n}{2} (n+1) + 2n$$

$$\sum_{r=1}^n (n+1)(n+2) = \frac{n}{3} (n^2 + 6n + 11)$$

EXERCISE 6: ALGEBRA

Evaluate the following

$$1. \sum_{r=1}^n n(n-1)$$

$$2. \sum_{r=1}^n (n+1)^2$$

$$3. \sum_{r=1}^n n^2(n+1)$$

$$4. \sum_{r=1}^n n(n+1)(n+2)$$

$$5. \sum_{r=1}^n (n+4)$$

$$6. \sum_{r=1}^n n^2(n-1)$$

$$7. \sum_{r=1}^n (n-1)(n+1)$$

$$8. \text{ Show that } \sum_{r=1}^n n^2 = \left(\sum_{r=1}^n n \right)^2$$

Write using sigma notation

$$9. (3 \times 7) + (5 \times 11) + (7 \times 15) + \dots + (13 \times 27)$$

$$10. 1 + 3 + 9 + \dots$$

$$11. 1^2 + 3^2 + 5^2 + \dots$$

$$12. 2^3 + 4^3 + 6^3 + \dots$$

6.5 Mathematical induction

Principle of mathematical induction states that “If $P(n)$ is any mathematical statement involving natural numbers n , $P(1)$ is true, if $P(k)$ is true, then $P(k+1)$ is true for all natural numbers”

Steps of proof by mathematical Induction

Let $P(n)$ be a proposition depending on a positive integer n .

Step 1: Show, if possible, that $P(n)$ is true for some positive integer n_0 .

Step 2: Show, if possible, that if $P(n)$ is true for an arbitrary integer $n = k \geq n_0$, then the proposition $P(k+1)$ follows from proposition $P(k)$

Step 3: If Step 2 is true, the fact that $P(n_0)$ is true implies that $P(n_0 + 1)$ is true, and then that $P(n_0 + 2)$ is true, and hence that $P(n)$ is true for all $n \geq n_0$

Step 4: If no number $n = n_0$ can be found for which Step 1 is true, or if in Step 2 it can be shown that $P(k)$ does not imply $P(k+1)$, the proposition $P(n)$ is false.

Example 41

Then $6^k = 7m - 8^k$ Prove by principle of mathematical induction that $6^n + 8^n$ is divisible by 7 for positive integers n

Solution

Step1: Let $P(n): 6^n + 8^n$ be a mathematical statement

Test: when $n = 1$: $P(1) = 6^1 + 8^1 = 14$, then $P(1)$ is true

Therefore, $P(n): 6^n + 8^n = 7m$ for $n \geq 1$

Induction hypothesis

$$P(k): 6^k + 8^k = 7m$$

Step 2: $n \geq k+1$

$$\begin{aligned} P(k+1): & 6^{k+1} + 8^{k+1} \\ &= 6^k 6 + 8^k 8 \\ &= 6(7m - 8^k) - 8^k \times 8 \\ &= 42m - 6 \times 8^k - 8^k \times 8 \\ \text{But } 6^k &= 7m - 8^k \\ &= 42m - (6 + 8)8^k \end{aligned}$$

$$\begin{aligned}
 &= 42m - 14 \times 8^k \\
 &= 7(6m - 2 \times 8^k)
 \end{aligned}$$

Hence shown that $6^n + 8^n$ has a factor of 7, therefore divisible by 7

Example 42

Prove that the sum of the first n positive integers is equal to $\frac{1}{2}n(n+1)$

Solution

Given $S_n : 1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$

Let the mathematical statement be $P(n) : 1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$

Test: $P(n) : 1 + 2 + 3 + \dots + 1 = \frac{1}{2} \times 2 = 1$, Then $P(1)$ is true.

Let $n = k \geq 1$

$P(k) : 1 + 2 + 3 + \dots + k = \frac{1}{2}k(k+1)$

Induction hypothesis

$P(k+1) : \underbrace{1 + 2 + 3 + \dots + k}_{P(k)} + (k+1)$

$P(k+1) : P(k) + (k+1)$

$P(k+1) : \frac{1}{2}k(k+1) + (k+1)$

$= \frac{1}{2}(k+1)(k+2)$ hence proved.

Example 43

Prove by the principle of mathematical induction that $2^n > n$ for $n \in \mathbb{Z}^+$

Solution

Let $P(n) : 2^n > n$ be a mathematical statements

Test $P(1) : 2^1 > 1$ therefore, $P(1)$ is true

For $n = k \geq 1$

$P(k) : 2^k > k$

Induction hypothesis

$P(k+1) : 2^{k+1} > k+1$

$$= 2^k \times 2^1 > k+1 \text{ (but } 2^k > k \text{)}$$

$$= k \times 2^1 = 2k > k+1$$

Hence shown that $P(k+1): 2k > k+1$

Example 44

Prove by mathematical induction that $\frac{d}{dx}(x^n) = nx^{n-1}$

Solution

Let the mathematical statement be $P(n): \frac{d}{dx}(x^n) = nx^{n-1}$

Test: if $n = 1$, $P(1): \frac{d}{dx}(x) = 1$, $P(1)$ is true

Let $n = k \geq 1: P(k): \frac{d}{dx}(x^k) = kx^{k-1}$

Induction hypothesis $P(k+1): \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x^k \times x)$

By product rule of differentiation $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$\frac{d}{dx}(x^k \times x) = x \frac{d}{dx}(x^k) + x^k \frac{d}{dx}(x)$$

$$\frac{d}{dx}(x^k \times x) = x(kx^{k-1}) + x^k(1)$$

$$\frac{d}{dx}(x^k \times x) = kx^k + x^k$$

$$\frac{d}{dx}(x^{k+1}) = (k+1)x^k \text{ Hence proved}$$

Example 45

Prove by the principle of mathematical induction that $x - y$ is a factor of $x^n - y^n$

Solution

Mathematical statement, $P(n): x^n - y^n$

Test: $P(1) = x^1 - y^1 = x - y$, $P(1)$ is true

Let $P(k): x^k - y^k = m(x - y)$

$$x^k = m(x-y) + y^k$$

Induction hypothesis: $P(k+1): x^{k+1} - y^{k+1}$

$$= x^k \times x - y^k \times y$$

$$= m(x-y)x + xy^k - y^k \times y$$

$$= mx(x-y) + y^k(x-y)$$

$$= (x-y)(mx + y^k)$$

Therefore, $P(k+1): (x-y)(mx + y^k)$ has a factor $x-y$

Example 46

Prove by mathematical induction that $\int x^n dx = \frac{x^{n+1}}{n+1}$

Solution

Mathematica statement: $P(n): \int x^n dx = \frac{x^{n+1}}{n+1}$

Test: $P(1): \int x dx = \frac{x^2}{2}$ then $P(1)$ is true

For $n = k \geq 1$, $P(k): \int x^k dx = \frac{x^{k+1}}{k+1}$

Induction hypothesis, $P(k+1): \int x^{k+1} dx = \int x^k \times x dx$

By using by-part method of integration $\int u dv = uv - \int v du$

$$\int x^k \times x dx = x \left(\frac{x^{k+1}}{k+1} \right) - \int \frac{x^{k+1}}{k+1} dx$$

$$\int (x^{k+1}) dx = \frac{x^{k+2}}{k+1} - \frac{1}{k+1} \left(\frac{x^{k+2}}{k+2} \right)$$

$$\int (x^{k+1}) dx = \frac{1}{(k+1)} \left(x^{k+2} - \frac{x^{k+2}}{k+2} \right)$$

$$\int (x^{k+1}) dx = \frac{1}{k+1} \left(\frac{(k+2)x^{k+2} - x^{k+2}}{k+2} \right)$$

$$\int (x^{k+1}) dx = \frac{(k+1)x^{k+2}}{(k+1)(k+2)} = \frac{x^{k+2}}{k+2} \text{ hence shown}$$

Example 47

Use mathematical induction to show that

$$\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\sin \frac{1}{2}(n+1)\theta \sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta}$$

Solution

Mathematical statement:

$$P(n) : \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin n\theta = \frac{\sin \frac{1}{2}(n+1)\theta \sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta}$$

Test: $n = 1$,

$P(1) : \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin \theta = \sin \theta$, $P(1)$ is true,

For $n = k \geq 1$,

$$P(k) : \sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin k\theta = \frac{\sin \frac{1}{2}(k+1)\theta \sin \frac{1}{2}k\theta}{\sin \frac{1}{2}\theta}$$

$$\begin{aligned} P(k+1) : & \underbrace{\sin \theta + \sin 2\theta + \sin 3\theta + \dots + \sin k\theta}_{P(k)} + \sin(k+1)\theta \\ &= \frac{\sin \frac{1}{2}(k+1)\theta \sin \frac{1}{2}k\theta}{\sin \frac{1}{2}\theta} + \sin(k+1)\theta \\ &= \sin(k+1)\theta = 2 \sin \frac{1}{2}(k+1)\theta \cos \frac{1}{2}(k+1)\theta \\ &= \frac{\sin \frac{1}{2}(k+1)\theta \sin \frac{1}{2}k\theta}{\sin \frac{1}{2}\theta} + 2 \sin \frac{1}{2}(k+1)\theta \cos \frac{1}{2}(k+1)\theta \\ &= \sin \frac{1}{2}(k+1)\theta \left(\frac{\sin \frac{1}{2}k\theta + 2 \cos \frac{1}{2}(k+1)\theta \sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta} \right) \end{aligned}$$

Factor formula: $\sin(A+B) - \sin(A-B) = 2 \cos A \sin B$

$$\begin{aligned}
&= \sin \frac{1}{2}(k+1)\theta \left(\frac{\sin \frac{1}{2}k\theta + \sin \frac{1}{2}(k+2)\theta - \sin \frac{1}{2}k\theta}{\sin \frac{1}{2}\theta} \right) \\
&= \frac{\sin \frac{1}{2}(k+1)\theta \sin \frac{1}{2}(k+2)\theta}{\sin \frac{1}{2}\theta} \text{ hence shown}
\end{aligned}$$

EXERCISE 7: ALGEBRA

1. State the principle of mathematical induction
2. Prove by mathematical induction that $9^n - 1$ is divisible by 8
3. Prove by induction that

$$(1 \times 1!) + (2 \times 2!) + (3 \times 3!) + \dots + (n \times n!) = (n+1)! - 1$$
4. If $x > 0$ show that $(1+x)^n > 1+nx$ for all n , $n \geq 2$
Prove the following by mathematical induction
5. $1+3+5+\dots+(2n-1)=n^2$
6. $1+4+7+\dots+(3n-2)=\frac{1}{2}n(3n-1)$
7. $1+5+9+\dots+(4n-3)=n(2n-1)$
8. $5+10+15+\dots+5n=\frac{5}{2}n(n+1)$
9. $6+12+18+\dots+6n=3n(n+1)$
10. $1^3+2^3+3^3+\dots+n^3=\frac{1}{4}n^2(n+1)^2$
11. $1^2+2^2+3^2+\dots+n^2=\frac{n}{6}(n+1)(2n+1)$
12. $1^2-2^2+3^2-\dots+(-1)^{n-1}(n^2)=\frac{1}{2}(-1)^{n-1}n(n+1)$
13. $1^2+3^2+5^2+\dots+(2n-1)^2=\frac{1}{3}n(4n^2-1)$
14. $2+2^2+2^3+\dots+2^n=2(2^n-1)$
15. $1 \times 2+2 \times 3+3 \times 4+\dots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$
16. $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

$$17. 1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$$

$$18. 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n^2 - 1)$$

Prove by mathematical induction that

$$19. 2^n \geq 1 + n \text{ for all } n \in \mathbb{Z}$$

$$20. n + 1 < 3n \text{ for all } n \in \mathbb{Z}$$

$$21. 1 + n^2 < (1 + n)^2 \text{ for all } n \in \mathbb{Z}$$

$$22. n < 1 + n^2 \text{ for all } n \in \mathbb{Z}$$

$$23. 7^n + 2 \text{ is divisible by } 3$$

$$24. n^3 + 5n \text{ is divisible by } 6$$

$$25. n(n^2 - 1)(3n + 2) \text{ is divisible by } 24$$

$$26. 7^n - 3^n \text{ is divisible by } 2 \text{ and } 4$$

$$27. n^5 - n \text{ is divisible by } 5$$

$$28. \sin(x + 180^\circ n) = (-1)^n \sin x, \text{ for all } n \in \mathbb{Z}$$

$$29. n! > n^2 \text{ for all } n \in \mathbb{Z}$$

30. By induction prove that the amount A which a principal P will accrue to at the end of n years if interest is compounded annually at the interest rate, r , is given by $A = P(1 + r)^n$

$$31. \text{ By induction prove that } \frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n-1}{n!} = 1 - \frac{1}{n!}$$

$$32. \text{ Use induction to prove that, if } y = \frac{1}{1-x} \text{ then } \frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$$

$$33. \text{ Show that } 2^n > n^2 \text{ for } n \geq 5$$

34. Using mathematical induction prove that

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

6.6 Roots of polynomial function

6.6.1 Quadratic equation

Polynomial of degree 2 is called **quadratic**.

Thus $ax^2 + bx + c = 0$ where $a \neq 0$ and a, b, c are constants

Solution of quadratic equations

Consider the equation $ax^2 + bx + c = 0$

Divide by a each term, $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, this can be written as

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Add $\left(\frac{b}{2a}\right)^2$ on both sides to make the left hand side a perfect square

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$ this is a general solution of quadratic equations.

The part $b^2 - 4ac$ is called a discriminant. This part determine the nature of the quadratic equation, there are three different cases

- When $b^2 - 4ac > 0$ the quadratic equation has two distinct real roots.
- When $b^2 - 4ac = 0$ the quadratic equation has two identical real roots.
- When $b^2 - 4ac < 0$ the quadratic equation has complex roots.

If the quadratic equation $ax^2 + bx + c = 0$ has roots (solutions) α and β

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Since α and β are roots we can use the roots to find the quadratic equation.

$$(x - \alpha)(x - \beta) = 0$$

$$x^2 - \alpha x - x\beta + \alpha\beta = 0 \Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Now, compare equations

$$\begin{cases} x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \\ x^2 - (\alpha + \beta)x + \alpha\beta = 0 \end{cases}$$

The sum of roots is $\alpha + \beta = -\frac{b}{a}$

Product of roots is $\alpha\beta = \frac{c}{a}$

Alternative

Since the roots of quadratic are $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and

$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$, thus the sum of roots is,

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ thus } \alpha + \beta = -\frac{b}{a}$$

The product of the roots is,

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right)$$

$$\alpha\beta = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}$$

Example 48

Find the sum and product of the roots of the following quadratic

(a) $3x^2 + 5x + 6 = 0$

(b) $x^2 + \frac{1}{2}x + 3 = 0$

(c) $\frac{3}{2}x^2 - 5x + 3 = 0$

Solution

(a) Given $3x^2 + 5x + 6 = 0$ thus $x^2 + \frac{5}{3}x + 2 = 0$

Sum of the roots: $\alpha + \beta = -\frac{b}{a} = -\frac{5}{3}$

Product of the roots: $\alpha\beta = \frac{c}{a} = 2$

(b) Given $x^2 + \frac{1}{2}x + 3 = 0$

The sum of roots: $\alpha + \beta = -\frac{1}{2}$

Product of roots: $\alpha\beta = \frac{c}{a} = 3$

(c) Given $\frac{3}{2}x^2 - 5x + 3 = 0$

$3x^2 - 10x + 6 = 0$

$x^2 - \frac{10}{3}x + 2 = 0$

The sum of the roots: $\alpha + \beta = \frac{10}{3}$

The product of the roots: $\alpha\beta = 2$

Example 49

Find the relation between a , b and c if one root of the quadratic $ax^2 + bx + c = 0$ is three times the other.

Solution

Given $ax^2 + bx + c = 0$

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

Given $\alpha = 3\beta$

Using $\alpha + \beta = 3\beta + \beta = -\frac{b}{a}$ (substitute $\alpha = 3\beta$)

Then $\beta = -\frac{b}{4a}$

From $\alpha\beta = \frac{c}{a}$ (substitute $\alpha = 3\beta$)

$$(3\beta)\beta = 3\beta^2 = \frac{c}{a} \Rightarrow \beta = \sqrt{\frac{c}{3a}}$$

Equate $-\frac{b}{4a} = \sqrt{\frac{c}{3a}}$

$$\left(-\frac{b}{4a}\right)^2 = \left(\frac{c}{3a}\right)$$

$$\frac{b^2}{16a^2} = \frac{c}{3a} \Rightarrow 3ab^2 = 16a^2c$$

The relation required, $3b^2 = 16ac$

Example 50

If α and β are the roots of the quadratic equation $x^2 - px + q = 0$ find the quadratic with roots $\frac{\alpha}{\beta^2}$ and $\frac{\beta}{\alpha^2}$

Solution

Given $x^2 - px + q = 0$ with roots α and β
 $\alpha + \beta = p$ and $\alpha\beta = q$

The new quadratic has roots $\frac{1}{\alpha\beta^2}$ and $\frac{1}{\alpha^2\beta}$

Sum of the roots of the new quadratic: $\frac{1}{\alpha\beta^2} + \frac{1}{\alpha^2\beta} = \frac{\alpha + \beta}{\alpha^2\beta^2} = \frac{p}{q^2}$

Product of the roots of the new quadratic: $\left(\frac{1}{\alpha\beta^2}\right)\left(\frac{1}{\alpha^2\beta}\right) = \frac{1}{(\alpha\beta)^3} = \frac{1}{q^3}$

The new equation is given by

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

Therefore the equation is $x^2 - \left(\frac{p}{q^2}\right)x + \frac{1}{q^3} = 0 \Rightarrow q^3x^2 - pqx + 1 = 0$

Example 51

If α and β are the roots of the quadratic equation $ax^2 + bx + a = 0$ find the new equation with the roots $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$

Solution

Given equation $ax^2 + bx + a = 0 \Rightarrow x^2 + \frac{b}{a}x + 1 = 0$

The sum of the roots: $\alpha + \beta = -\frac{b}{a}$

The product of the roots: $\alpha\beta = -1$

The new equation has sum of roots:

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{\alpha^3\beta^3} = \frac{(\alpha + \beta)(\alpha + \beta)^2 - 3\alpha\beta}{(\alpha\beta)^3}$$

$$\left(\frac{b}{a}\right)\left(\left(-\frac{b}{a}\right)^2 + 3\right)$$

Sum of the roots: $\frac{b^3}{a^3} + \frac{3b}{a}$

Product of roots of the new equation: $\frac{1}{\alpha^3} \times \frac{1}{\beta^3} = \frac{1}{(\alpha\beta)^3} = -1$

The equation is given by $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

The new equation is: $x^2 - \left(\frac{b^3}{a^3} + \frac{3b}{a}\right)x - 1 = 0$

$$\Rightarrow a^3x^2 - (b^3 + 3a^2b)x - a^3 = 0$$

Questions

- If one root of the equation $x^2 + bx + c = 0$ is the square of the other, show that $b^3 + 2c^2 = 3bc$
- If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $m : n$ find the equation connecting a, b, c, m and n .
- Find the condition that one root of the quadratic equation $px^2 - qx + p = 0$ may be three more than the other.

EXERCISE 8: ALGEBRA

- One roots of the equation $ax^2 + bx + c = 0$ is the twice of the other, show that $b = 3\sqrt{\frac{ac}{2}}$
- The roots of the quadratic $x^2 - 4ax + 2b = 0$ are α and β . If it has equal roots show that $b - 2a^2 = 0$
- If α and β are the roots of the equation $x^2 + e^2x + 2 = 0$ find the equation with roots $\alpha + 1$ and $\beta + 1$
- Find the relation between a and b if one root of the equation $x^2 - 4ax + 2b = 0$ is 4 times the other.
- Find the new equation with roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ if the equation $3x^2 - 4x + 1 = 0$ has roots α and β .
- The roots of the quadratic equation $2x^2 + 8x - 6 = 0$ are α and β find
 - $\alpha - \beta$
 - $\alpha^3 - \beta^3$
 - $\alpha^2 - \beta^2$
 - $\alpha^3 + \beta^3$
- If the equations $ax^2 + 2bx - 3c = 0$ and $x^2 + bx + c = 0$ have common root, find the relation connecting a , b and c .
- The roots of $x^2 + px + 5 = 0$ are α and β , if $\alpha^2 + \beta^2 - 3 = 0$, show that $p^2 = 13$
- The roots of the quadratic equation $x^2 - 3x + 1 = 0$ are α and β , without solving find the quadratic equation with roots $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$
- Simplify the expression $\left(\frac{1+x^2}{1-x^2}\right)^2 - \left(\frac{2x}{1-x^2}\right)^2$

6.6.1 Cubic equation

An equation $ax^3 + bx^2 + cx + d = 0$ is a cubic equation, where a, b, c and d are constants, where $a \neq 0$

Let α, β and γ be the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$.
Then

$$ax^3 + bx^2 + cx + d = 0 \Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

Since α, β and γ are the roots of the equation, then
 $(x - \alpha)(x - \beta)(x - \gamma) = 0$

$$x^3 - \alpha x^2 - \beta x^2 + \alpha \beta x - \gamma x^2 + \alpha \gamma x + \beta \gamma x - \alpha \beta \gamma = 0$$

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$$

Compare

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

Then, sum of roots: $(\alpha + \beta + \gamma) = -\frac{b}{a}$

Sum-product of roots: $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

Product of roots: $\alpha\beta\gamma = -\frac{d}{a}$

Example 52

If α, β and γ are roots of the cubic equation $x^3 + ax^2 + bx + c = 0$, find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$.

Solution

From $x^3 + ax^2 + bx + c = 0$

The roots sum of the roots: $\alpha + \beta + \gamma = -a$

The sum-product of the roots: $\alpha\beta + \beta\gamma + \alpha\gamma = b$

The product of roots: $\alpha\beta\gamma = -c$

Roots of the new equation

Sum of the roots $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = -\frac{b}{c}$

Sum-product of the roots: $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} = \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma} = -\frac{a}{c}$

Product of the roots: $\frac{1}{\alpha} \times \frac{1}{\beta} \times \frac{1}{\gamma} = \frac{1}{\alpha\beta\gamma} = -\frac{1}{c}$

The new equation is $x^3 - (\text{sum})x^2 + (\text{sumproduct})x + (\text{product}) = 0$

$$x^3 + \left(\frac{b}{c}\right)x^2 - \left(\frac{a}{c}\right)x - \left(\frac{1}{c}\right) = 0 \text{ thus } cx^3 + bx^2 - ax - 1 = 0$$

Example 53

The roots of the equation $4x^3 + 3x^2 - 2x + 5 = 0$ find without solving, the new equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

Solution

The old equation is an equation of x ,

Let new equation be the equation of y

The old equation has roots $x = \{\alpha, \beta, \gamma\}$

The new equation has roots $y = \left\{\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right\}$

The relation between y and x is $y = \frac{1}{x}$

Make x the subject and substitute in the old equation to get the new equation in terms of y .

$$x = \frac{1}{y}$$

$$\begin{aligned} \text{Then, } 4x^3 + 3x^2 - 2x + 5 = 0 &\Rightarrow 4\left(\frac{1}{y}\right)^3 + 3\left(\frac{1}{y}\right)^2 - 2\left(\frac{1}{y}\right) + 5 = 0 \\ &= \frac{4}{y^3} + \frac{3}{y^2} - \frac{2}{y} + 5 = 0 \end{aligned}$$

The new equation is, $4 + 3y - 2y^2 + 5y^3 = 0$ to change the equation to be in term of x , replace y with x , the equation become,

$$5x^3 - 2y^2 + 3y + 4 = 0$$

Example 54

Find the relation between the constants a , b , c and d if the roots of the equation $ax^3 + bx^2 + cx + d = 0$ are arithmetic progression.

Solution

$$\text{Sum of roots: } \alpha + \beta + \gamma = -\frac{b}{a}$$

$$\text{Sum-product of roots: } \alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$$

$$\text{Product of roots: } \alpha\beta\gamma = -\frac{d}{a}$$

$$\alpha + \beta + \gamma$$

$$d = \beta - \alpha = \gamma - \beta$$

$$\beta = \frac{\alpha + \gamma}{2}$$

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\frac{3\alpha + 3\gamma}{2} = -\frac{b}{a} \Rightarrow \alpha + \gamma = -\frac{2b}{3a}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$$

$$\alpha\left(\frac{\alpha + \gamma}{2}\right) + \gamma\left(\frac{\alpha + \gamma}{2}\right) + \alpha\gamma = \frac{c}{a}$$

$$\left(\frac{\alpha + \gamma}{2}\right)(\alpha + \gamma) + \alpha\gamma = \frac{c}{a}$$

$$2(\alpha + \gamma)(\alpha + \gamma) + 2\alpha\gamma = \frac{2c}{a}$$

$$\left(-\frac{2b}{3a}\right)\left(-\frac{2b}{3a}\right) + 2\alpha\gamma = \frac{2c}{a}$$

$$\frac{4b^2}{9a^2} + 2\alpha\gamma = \frac{2c}{a}$$

$$\alpha\gamma = \left(\frac{c}{a} - \frac{2b^2}{9a^2}\right)$$

Product of roots $\alpha\beta\gamma = -\frac{d}{a}$

$$\alpha\gamma\left(\frac{\alpha+\gamma}{2}\right) = -\frac{d}{a}$$

$$\alpha\gamma\left(-\frac{b}{3a}\right) = -\frac{d}{a}$$

$$\frac{b}{3a}\alpha\gamma = \frac{d}{a} \text{ therefore } \alpha\gamma = \frac{3d}{b}$$

Compare, $\frac{3d}{b} = \frac{c}{a} - \frac{2b^2}{9a^2}$

$$3d = \frac{bc}{a} - \frac{2b^2}{9a^2}$$

The relation required is $27a^2d = 9abc - 2b^3$

6.7 Identities

$$(a) \quad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$(b) \quad a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$(c) \quad a^3 + b^3 + c^3 = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac) + 3abc$$

EXERCISE 9: ALGEBRA

1. The equation $3x^3 + 6x^2 - 4x + 7 = 0$ has roots α , β and γ . Find the equation with roots

$$(a) \quad \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$$

$$(b) \quad \beta + \gamma, \alpha + \gamma \text{ and } \alpha + \beta$$

$$(c) \quad \alpha^2, \beta^2 \text{ and } \gamma^2$$

2. The equation $4x^3 + 2x^2 + 3x + 6 = 0$ has roots α , β and γ . Find the equation with roots

$$(a) \quad \alpha^2, \beta^2 \text{ and } \gamma^2$$

$$(b) \quad \alpha - 2, \beta - 2 \text{ and } \gamma - 2$$

(c) $2\alpha+1$, $2\beta+1$ and $2\gamma+1$

(d) $\frac{1}{\alpha\beta}$, $\frac{1}{\beta\gamma}$ and $\frac{1}{\alpha\gamma}$

3. If the roots of the equation $x^3 - qx + r = 0$ are α , β and γ , show that $\alpha^2 + \beta^2 + \gamma^2 = q^2$
4. One of the roots of the equation $21x^3 - 50x^2 - 37x - 6 = 0$ is a positive integer. Find this root and hence solve the equation completely
5. Show that, if the roots of the equation $x^3 - 5x^2 + ax - 8 = 0$ are in geometric progression, then $a = 10$
6. Solve $12x^3 - 52x^2 + 35x + 50 = 0$ if the equation has repeated roots.

6.8 Remainder theorem and division of polynomials

The operation of division of polynomials is analogous to the operation of

division of integers. $\frac{\text{Dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$

$\Rightarrow \text{Divident} = (\text{quotient}) \times (\text{divisor}) + \text{remainder}$

Example 55

Divide $x^3 + 6x^2 - 2x - 5$ by $x + 5$ and state the remainder

Solution

Using long – division method

$$\begin{array}{r}
 x^2 + x - 7 \\
 x+5 \overline{) x^3 + 6x^2 - 2x - 5} \Rightarrow \text{The remainder is 30} \\
 \underline{x^3 + 5x^2} \\
 x^2 - 2x \\
 \underline{x^2 + 5x} \\
 -7x - 5 \\
 \underline{-7x - 35} \\
 30
 \end{array}$$

If we divide a polynomial $P(x)$ by the linear polynomial $D(x) = x - a$, then the division algorithm states that there exist a unique polynomial $Q(x)$

and $R(x)$ such that $P(x) = D(x)Q(x) + R(x) \Rightarrow P(x) = (x - a)Q(x) + R(x)$.
When $x = a$ then $P(a) = (a - a)Q(a) + R(a) \Rightarrow P(a) = R(a)$.

The remainder theorem states that "If a polynomial $P(x)$ is divided by $x - a$, then the remainder is $P(a)$ "

Consider example 55 above, using the remainder theorem we can easily find the remainder as follows,

The polynomial $P(x) = x^3 + 6x^2 - 2x - 5$ is divided by $D(x) = x + 5$ using the theorem, $D(x) = 0$

$$x + 5 = 0 \Rightarrow x = -5$$

$$P(5) = (-5)^3 + 6(-5)^2 - 2(-5) - 5 = 30$$

Note that, when a polynomial is divisible by another polynomial the remainder is zero.

In other words, if a polynomial is divided by $x - a$ and the remainder is zero, then $x - a$ is a factor of the polynomial.

Example 56

Find the remainder when $5x^3 - 3x^2 + 4x + 2$ is divided by $x - 3$

Solution

Using the remainder theorem

$$x - 3 = 0 \Rightarrow x = 3$$

$$P(x) = 5x^3 - 3x^2 + 4x + 2$$

$$P(3) = 5(3)^3 - 3(3)^2 + 4(3) + 2 = 122$$

The remainder is 122.

Example 57

Find the remainder when the polynomial $x^3 - 3x^2 - 7x + 9$ is divided by $x + 2$

Solution

$$x + 2 = 0 \Rightarrow x = -2$$

The polynomial is $x^3 - 3x^2 - 7x + 9$

$$P(-2) = (-2)^3 - 3(-2)^2 - 7(-2) + 9 = 3$$

6.9 Synthesis division method

Synthesis division is an efficient method for dividing polynomials by another polynomial, this method gives the remainder and the quotient.

Example 58

- Divide $4x^3 + 6x^2 + 3x + 2$ by $x - 3$
- Find the quotient and the remainder when $x^3 + 4x^2 - 10x - 3$ is divided by $x + 4$
- Find the quotient and remainder when $x^4 + 3x^3 - 4x^2 + 3x - 4$ is divided by $x - 5$

Solution

(a) By synthesis method

List the coefficients of the dividend, write them in a line, and write the divisor value at the left end of all other coefficient.

Procedures

- Take the first coefficient, (4) down the bottom line (shown below)
- Take a divisor, (3) value times the dropped number (4) and write the answer below the next coefficient, (6) and add the two numbers to get 18, write it below the line next to 4.
- Repeat the procedures to the end of coefficients. (See below)

$$\begin{array}{r|rrrr}
 3 & 4 & 6 & 3 & 2 \\
 + & \downarrow & 12 & 54 & 171 \\
 \hline
 & 4 & 18 & 57 & 173
 \end{array}$$

← Remainder

Obtained below are the coefficient of the quotient polynomial, the last number is a remainder. Remember when dividing polynomials the quotient will be a polynomial less than divided polynomial.

$$\text{Thus } \frac{4x^3 + 6x^2 + 3x + 2}{x - 3} = 4x^2 + 18x + 57 + \frac{173}{x - 3}$$

The remainder is 173, the quotient is $4x^2 + 18x + 57$ see (b) and (c) below

(b) $x^3 + 4x^2 - 10x - 3$ the divisor $x + 4$

-4	1	4	-10	-3	Coefficients of the polynomial
	+	-4	0	40	
	1	0	-10	37	Remainder

Coefficients degree 2 polynomial

The quotient is $x^2 - 10$ and the remainder is 37

(c) $x^4 + 3x^3 - 4x^2 + 3x - 4$ divisor $x - 5$

5	1	3	-4	3	-4	Coefficients of the polynomial
	+	5	40	180	915	
	1	8	36	183	911	Remainder

Coefficients Degree 3 polynomial

The quotient is $x^3 + 8x^2 - 36x + 183$ the remainder is 911

Note that synthesis division can not be used to divide polynomial of degree n with another polynomial of degree greater than 1. Thus dividing $3x^2 + 4x^3 - 10x - 15$ by $x^2 + 3x + 3$ can be done by long division. In other words, synthesis division is used to divide polynomials with a linear polynomial.

EXERCISE 10: ALGEBRA

1. Find the relation between q and r so that $x^3 + 3px^2 + qx + r$ is a perfect cube for all values of x .
2. Find the remainder when the polynomial $6x^3 + x^2 + 13x + 12$ is divided by $3x + 2$
3. When the polynomial $ax^3 + bx^2 + cx - 6 = 0$ is divisible by $x + 2$, $x - 3$ and $2x + 1$ find the values of a , b and c .
4. If $x + 3$ is a factor of $x^3 - 9x + k$ find the value of k .
5. Find the remainder when the polynomial $3x^3 - 8x^2 + 8x - 5$ is divided by $x - 1$

6. If the polynomial $2x^3 - ax^2 + bx + 5 = 0$ is exactly divisible by $x - 2$ and leaves a remainder of 10 when divided by $x + 3$, find the values of a, b .

6.10 INEQUALITIES

An inequality is a mathematical statement which shows that one side of is either less or greater than the other side. Inequalities can also be solved using graphs by shading the region that satisfy the inequality.

Solving inequalities

(a) Inequalities of the form $ax^2 + bx + c < 0$ or $ax^2 + bx + c > 0$

Example 59

Find the set of values of x for which $2x^2 + 5x - 3 > 0$

Solution

Given $2x^2 + 5x - 3 = (2x - 1)(x + 3) > 0$

For the function $2x^2 + 5x - 3$ to be positive, either $(2x - 1) > 0$ and $(x + 3) > 0$ or $(2x - 1) < 0$ and $(x + 3) < 0$

Solve for x from the factors above, $x = -3$ and $x = 1/2$

Now think of numbers less than -3 , numbers between -3 and $1/2$ and numbers greater than $1/2$ and tabulate as shown below

	$x < -3$	$-3 < x < 1/2$	$x > 1/2$
$x + 3$	-ve	+ve	+ve
$2x - 1$	-ve	-ve	+ve
$(2x - 1)(x + 3)$	+ve	-ve	+ve

Thus to satisfy the inequality $2x^2 + 5x - 3 > 0$ the set of values of x are $x < -3$ or $x > 1/2$

Example 60

Find the set of values for which $1 + 2x - 3x^2 < 0$

Solution

Solve the inequality as an equation $1 + 2x - 3x^2 = 0$

$(3x+1)(x-1)=0$ solution are $x = -\frac{1}{3}$ and $x = 1$

	$x < -1/3$	$-1/3 < x < 1$	$x > 1$
$(3x+1)$	-ve	+ve	+ve
$(x-1)$	-ve	-ve	+ve
$(3x+1)(x-1)$	+ve	-ve	+ve

To satisfy the condition of the inequality $1+2x-3x^2 < 0$ then $-1/2 < x < 1$

Example 61

Find the set of values for which $x^2 + 2x - 3 < 0$ is negative?

Solution

Solve the inequality $x^2 + 2x - 3 < 0$ as an equation $x^2 + 2x - 3 = 0$
factors $(x-1)(x+3)=0$

$x = 1$ and $x = -3$

	$x < -3$	$-3 < x < 1$	$x > 1$
$(x-1)$	-ve	-ve	+ve
$(x+3)$	-ve	+ve	+ve
$-\frac{2x+3}{x+2} < 0$	+ve	-ve	+ve

To satisfy the inequality, $x^2 + 2x - 3 < 0$ the solution is $-3 < x < 1$

(b) Rational inequality

In this section we shall be concerned with rational functions of two

quadratic or linear that is function of the form $\frac{ax^2 + bx + c}{px^2 + qx + r}$ or $\frac{ax + b}{cx + d}$

Example 62

Find the set of values of x for which $\frac{4x+8}{x-1} > 3$

Solution

Make the right-hand-side zero

$$\frac{4x+8}{x-1} - 3 > 0 \text{ thus } \frac{x+11}{x-1} > 0$$

Solve for x from the numerator and denominator

$$x+11=0 \Rightarrow x = -11 \text{ and } x-1=0 \Rightarrow x = 1$$

	$x < -11$	$-11 < x < 1$	$x > 1$
$x + 11$	-ve	+ve	+ve
$x - 1$	-ve	-ve	+ve
$\frac{x + 11}{x - 1}$	+ve	-ve	+ve

The solution is $x < -11$ or $x > 1$

Example 63

Find the set of values of x for which $\frac{2x+5}{x+2} < 4$

Solution

$$\frac{2x+5-4x-8}{x+2} < 0 \text{ thus } -\frac{2x+3}{x+2} < 0$$

Divide by the negative sign

$$\frac{2x+3}{x+2} > 0 \text{ then } x = -1.5 \text{ and } x = -2$$

	$x < -2$	$-2 < x < -1.5$	$x > -1.5$
$2x + 3$	-ve	-ve	+ve
$x + 2$	-ve	+ve	+ve
$\frac{2x + 3}{x + 2}$	+ve	-ve	+ve

The solution set is $x < -2$ or $x > -1.5$

(c) Absolute values inequalities

The absolute value of x is denoted as $|x|$ is always positive and it is

$$\text{defined as } |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Consider the an absolute function $|ax + b| < c$

Square both sides $(|ax + b|)^2 < c^2$

$$ax^2 + 2abx + b^2 < c^2$$

$ax^2 + 2abx + b^2 - c^2 < 0$, this is a quadratic inequality and can be solve using the previous method.

Example 64

Find the set of real values of x for which $|3x + 1| < 2$

Solution

Given $(|3x + 1|)^2 < 2^2$
 $9x^2 + 6x + 1 < 4$
 $9x^2 + 6x - 3 < 0$
 $(3x - 1)(x + 1) < 0$
 $x = 1/3$ or $x = -1$

	$x < -1$	$-1 < x < 1/3$	$x > 1/3$
$(3x - 1)$	-ve	-ve	+ve
$(x + 1)$	-ve	+ve	+ve
$(3x - 1)(x + 1)$	+ve	-ve	+ve

The solution is $-1 < x < 1/3$

EXERCISE 11: ALGEBRA

Find the set values of x for each of the following inequality

1. $x^2 + 4x + 3 > 0$
2. $3x^2 - 5x + 2 < 0$
3. $\frac{4x + 8}{x - 1} > 3$
4. $\frac{3 + 4x}{x} > 3$
5. $\frac{x^2 - 3x + 2}{3x^2 - 4x + 1} < 0$
6. $\frac{3 - 4x}{2 - x} < 2$
7. $\frac{1 - 4x}{x - 2} > 4$
8. $|x - 3| > 4$
9. $|x + 2| < 1$
10. $|2x + 5| > 3$
11. $|x - 3| < 4$
12. $|5 - x| < 2$
13. $|1 - x| > 3$
14. $|2 + x| > |1 + 2x|$
15. $|5 - x| < |1 + x|$
16. $|1 - 2x| \leq |3x - 1|$
17. $|4x - 2| \geq |2 - 3x|$
18. $|x + 2| \geq |2x - 1|$

$$19. \frac{2x+3}{2x-1} > 5$$

$$20. 2x-1 < x^2-4 < 12$$

$$21. \frac{x+4}{2x-1} > 5$$

6.11 Matrices and determinants

English mathematician Cayley introduced matrices in 1860's. Matrices become greater utility in many branches of mathematics.

Matrix: Is the rectangular array of numbers bounded by brackets. Matrices are denoted by capital letters, $A, B, C, D \dots$

The order of the matrix is the dimension of the matrix, which is determined by order = rows $\times$ columns

(a) Example the order of matrix $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$ is 3×2 which reads "3 by 2" mean 3 rows and 2 columns.

(b) Example the order of matrix $B = \begin{pmatrix} 2 & 3 & 4 \\ 6 & 5 & 8 \\ 5 & 8 & 1 \end{pmatrix}$ is 3×3 matrix

Types of matrices

1. Row matrix: Is a matrix with only one row, i.e. $A = (2 \quad 4 \quad -6)$

2. Column matrix: is a matrix with only one column, i.e. $B = \begin{pmatrix} 6 \\ 7 \\ 2 \end{pmatrix}$

3. Square matrix: Is a matrix with the equal number of column and rows. Example: $C = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$

4. Zero/null matrix: is a matrix with all elements zero. Example:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

5. Diagonal matrix: a square matrix in which all non-diagonal (main

diagonal) elements are zero. Example: $E = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

6. Identity matrix: Is a diagonal matrix with all diagonal elements are one (1). It is denoted by I .

Example: $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

7. Scalar matrix: is a diagonal matrix, whose all diagonal elements are

equal. Example: $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Transpose of a matrix

In a matrix, when columns and rows are interchanged so that rows become columns the new matrix formed is called a **transpose**.

Example: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then A transpose is denoted as A^T , this is

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Questions

Find a transpose of each of the following:-

1. $A = \begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix}$

2. $B = \begin{pmatrix} -3 & 5 \\ 10 & 1 \end{pmatrix}$

3. $C = \begin{pmatrix} 3 & 4 & -4 \\ 2 & 0 & 6 \\ 7 & 9 & 1 \end{pmatrix}$

6.11.1 Matrices operations

Addition and subtraction

Addition or subtraction of the matrices is possible only if the matrices are of the same order. If A and B are two matrices of the same order, then their sum or difference ($A \pm B$) is defined as the new matrix in which the elements are the sum of the corresponding elements of A and B . If $A = (a_{ij})$ and $B = (b_{ij})$ then $A + B = (a_{ij} + b_{ij})$ for $\forall ij$ where i is columns and j is rows.

Example 65

If $A = \begin{pmatrix} 2 & 7 \\ 8 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 8 \\ 3 & 3 \end{pmatrix}$ determine (a) $A + B$ (b) $A - B$

Solution

$$(a) \quad A + B = \begin{pmatrix} 2 & 7 \\ 8 & 1 \end{pmatrix} + \begin{pmatrix} 6 & 8 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 15 \\ 11 & 4 \end{pmatrix}$$

$$(b) \quad A - B = \begin{pmatrix} 2 & 7 \\ 8 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 8 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ 5 & -2 \end{pmatrix}$$

Note that, matrix addition is commutative, that is $A + B = B + A$ and $A - B \neq B - A$ where A and B are two matrices of the same order.

Example 66

Given $A = \begin{pmatrix} 7 & -8 & 1 \\ 6 & 7 & -1 \\ 2 & -3 & -8 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 9 & 2 \\ -2 & -1 & 3 \\ 2 & 4 & 5 \end{pmatrix}$ find

(a) $A + B$ (b) $A - B$

Solution

$$(a) \quad A + B = \begin{pmatrix} 7 & -8 & 1 \\ 6 & 7 & -1 \\ 2 & -3 & -8 \end{pmatrix} + \begin{pmatrix} 1 & 9 & 2 \\ -2 & -1 & 3 \\ 2 & 4 & 5 \end{pmatrix}$$

$$A+B=\begin{pmatrix} 8 & 1 & 3 \\ 4 & 6 & 2 \\ 4 & 1 & -3 \end{pmatrix}$$

$$(b) \quad A-B=\begin{pmatrix} 7 & -8 & 1 \\ 6 & 7 & -1 \\ 2 & -3 & -8 \end{pmatrix}-\begin{pmatrix} 1 & 9 & 2 \\ -2 & -1 & 3 \\ 2 & 4 & 5 \end{pmatrix}$$

$$A-B=\begin{pmatrix} 6 & -17 & -1 \\ 8 & 8 & -4 \\ 0 & -7 & -13 \end{pmatrix}$$

Question

Prove the associative law of matrices, thus

$$(a) \quad A+(B+C)=(A+B)+C$$

$$(b) \quad \text{Show that } A-(B-C)\neq(A-B)-C$$

Additive identity of matrices

Let A and B be any two matrices of order $m \times n$, if A is a non-zero matrix and $A+B=B+A=A$ then B is called additive inverse of A , and B is a zero matrix.

If $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $A+B=B+A=A$

Additive inverse of matrices

Let A and B be any two matrices of order $m \times n$, if A and B are non-zero matrices and $A+B=B+A=O$ where O is a zero matrix, then A is called additive inverse of B and vice-versa.

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, if $A+B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} p & q \\ r & s \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then

A is additive inverse of B and vice versa.

Scalar multiplication of matrices

If a matrix of any order $m \times n$ is multiplied by a scalar quantity k , then all elements of the matrix are multiplied by the scalar k .

That is, if $A = \begin{pmatrix} 2 & -3 & 4 \\ -4 & 7 & -6 \\ 8 & 2 & -1 \end{pmatrix}$ then

$$kA = k \begin{pmatrix} 2 & -3 & 4 \\ -4 & 7 & -6 \\ 8 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 2k & -3k & 4k \\ -4k & 7k & -6k \\ 8k & 2k & -1k \end{pmatrix}, \text{ note that } k \text{ is a number, not a}$$

matrix.

Conversely: If the matrix elements have the common multiple, the common number (multiple) can be factorized from each element of the matrix

That is if $A = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$ thus $A = k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Note that, $k(A \pm B) = kA \pm kB$

Example 67

If $A = \begin{pmatrix} 5 & -2 & 2 \\ 1 & 3 & -7 \\ 8 & -2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 7 & 2 \\ 4 & 3 & -2 \\ -3 & 2 & -1 \end{pmatrix}$ evaluate $2A + 3B$

Solution

$$\begin{aligned} 2A + 3B &= 2 \begin{pmatrix} 5 & -2 & 2 \\ 1 & 3 & -7 \\ 8 & -2 & 6 \end{pmatrix} + 3 \begin{pmatrix} 6 & 7 & 2 \\ 4 & 3 & -2 \\ -3 & 2 & -1 \end{pmatrix} \\ 2A + 3B &= \begin{pmatrix} 10 & -4 & 4 \\ 2 & 6 & -14 \\ 16 & -4 & 12 \end{pmatrix} + \begin{pmatrix} 18 & 21 & 6 \\ 12 & 9 & -6 \\ -9 & 6 & -3 \end{pmatrix} \\ 2A + 3B &= \begin{pmatrix} 28 & 17 & 10 \\ 14 & 15 & -20 \\ 7 & 2 & 9 \end{pmatrix} \end{aligned}$$

Example 68

Given $A + B = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$ and $A - B = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix}$ find matrices A and B

Solution

$$+ \begin{cases} A+B = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} \\ A-B = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} \end{cases}$$

$$2A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 5 & 9 \end{pmatrix} \text{ thus } A = \begin{pmatrix} 1 & 7/2 \\ 5/2 & 9/2 \end{pmatrix}$$

$$- \begin{cases} A+B = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} \\ A-B = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} \end{cases}$$

$$2B = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ thus } B = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Question

Show that $3(A+B) = 3A + 3B$

Multiplication of matrices

Multiplication of two matrices A and B is possible only if the number of columns in pre-multiplier, A , (first) matrix is equal to the number of rows in post-multiplier, B , (second) matrix.

Example 69

Find $A \times B$ in $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $B = \begin{pmatrix} x \\ y \end{pmatrix}$

Solution

$$A \times B = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax + cy \quad bx + dy)$$

Example 70

Find $A \times B$ if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ k & r \end{pmatrix}$

Solution

$$A \times B = AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ k & r \end{pmatrix}$$

$$AB = \begin{pmatrix} ax + bk & ay + br \\ cx + dk & cy + dr \end{pmatrix}$$

Multiplication of 3 by 3 matrices

Consider the matrices $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ and $B = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$

The product $A \times B = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \times \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$ is

$$A \times B = \begin{pmatrix} a_1x_1 + a_2y_1 + a_3z_1 & a_1x_2 + a_2y_2 + a_3z_2 & a_1x_3 + a_2y_3 + a_3z_3 \\ b_1x_1 + b_2y_1 + b_3z_1 & b_1x_2 + b_2y_2 + b_3z_2 & b_1x_3 + b_2y_3 + b_3z_3 \\ c_1x_1 + c_2y_1 + c_3z_1 & c_1x_2 + c_2y_2 + c_3z_2 & c_1x_3 + c_2y_3 + c_3z_3 \end{pmatrix}$$

Example 71

Evaluate AB if $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 8 & 9 \\ 0 & 2 \end{pmatrix}$

Solution

$$AB = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 2 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 8 & 9 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 \times 3 + 1 \times 8 + 2 \times 0 & 0 \times 4 + 1 \times 9 + 2 \times 2 \\ 1 \times 3 + 2 \times 8 + 3 \times 0 & 1 \times 4 + 2 \times 9 + 3 \times 2 \\ 3 \times 3 + 2 \times 8 + 4 \times 0 & 3 \times 4 + 2 \times 9 + 4 \times 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 8 & 13 \\ 18 & 28 \\ 22 & 38 \end{pmatrix}$$

Why BA in this case is not defined?

Properties of matrix multiplication

1. Matrix multiplication is not commutative, that is $AB \neq BA$
2. Matrix multiplication is associative, that is $A(BC) = (AB)C$
3. Let I be identity matrix, A be any matrix of the same order as I , then $AI = IA = A$
4. Let A be the inverse of matrix B , then $AB = I$, where I is Identity matrix

Orthogonal matrix

A matrix is said to be orthogonal matrix if $A \times A^T = I$ where I is an identity matrix, A is any matrix of order $m \times n$ and A^T is a transpose of A .

Example 72

If $A = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix}$ show that A is orthogonal.

Solution

$$A = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix} \text{ and } A^T = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

$$A \times A^T = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{pmatrix} \times \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{pmatrix}$$

$$A \times A^T = \frac{1}{9} \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, A is orthogonal matrix.

Note that:

If a singular matrix is multiplied by another matrix (not necessary singular), the resulting matrix is a singular matrix.

Proof: Let the matrix A be $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$ and let matrix B be $\begin{pmatrix} x & y \\ x & y \end{pmatrix}$ which are singular matrices, then $A+B = \begin{pmatrix} a & b \\ a & b \end{pmatrix} + \begin{pmatrix} x & y \\ x & y \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ a+x & b+y \end{pmatrix}$ therefore $|A+B| = (a+x)(b+y) - (a+x)(b+y) = 0$

Question

Use $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ x & y \end{pmatrix}$ show that the following matrices has no inverses.

- (i) $A+3B$
- (ii) $2A+3B$
- (iii) AB
- (iv) $AB+2B$

EXERCISE 12: ALGEBRA

1. Determine (i) $2A + B$ (ii) $3A - 2B$ if

(a) $A = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 7 \\ 9 & 2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 2 & 1 \\ -3 & -1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 7 & 1 \\ 2 & 0 & 0 \\ 2 & 1 & 3 \end{pmatrix}$

2. If $A = \begin{pmatrix} 2 & 1 & -3 \\ 2 & -2 & 1 \\ 3 & -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & -2 \\ 3 & -4 & 0 \\ 3 & -1 & 3 \end{pmatrix}$, show that $(AB)^T = B^T A^T$

3. Find (i) AB (ii) BA if $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -4 & 5 \\ 0 & -2 & 1 \\ 6 & -2 & -3 \end{pmatrix}$

4. If $2 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 4 \\ 2 & 3w \end{pmatrix} + \begin{pmatrix} 5 & x-y \\ z+2w & 3w-1 \end{pmatrix}$ find x, y, z and w

5. Given that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ find (i) AB (ii) BA

Determinant of 3 by 3 matrices

Only a square matrix has a determinant, (a reader should be familiar with the determinant of 2 by 2 matrices). Simply the determinant of 2 by 2 matrix is obtained by taking *the product of elements of the leading diagonal minus the product of the elements of the main diagonal*.

Determinant is denoted by absolute value of the matrix name, $|A|$ or by Δ . Before we find the determinant, we shall first define the matrix direction sign, these signs goes with the respective element of the particular position when finding the determinant.

The matrix direction sign are defined as $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Here we shall discuss two ways to obtain the determinant of 3 by 3 matrix

Suppose the matrix is $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, the determinant of A is obtained

as follows:- $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ taking the first row into consideration, all the

first row elements become scalar factors of minor determinants formed by excluding the elements of the row and column containing the scalar.

$|A| = +a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$. This depict that a 3 by 3 matrix is

now decomposed to small determinants of 2 by 2 with a scalar outside them. To obtain the determinant, obtain evaluate the small determinants.

Example 73

Find the determinant of the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -2 & 4 \\ 0 & 3 & 2 \end{pmatrix}$

Solution

Given $A = \begin{pmatrix} 1 & 0 & 3 \\ 3 & -2 & 4 \\ 0 & 3 & 2 \end{pmatrix}$

$$|A| = +1 \begin{vmatrix} -2 & 4 \\ 3 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + 3 \begin{vmatrix} 3 & -2 \\ 0 & 3 \end{vmatrix}$$

Therefore, the determinant is $|A| = 11$

When a determinant of matrix is zero the matrix is said to be a singular matrix.

Example 74

Find the determinant of matrix $A = \begin{pmatrix} 2 & 3 & 1 \\ -3 & 2 & -4 \\ 4 & 6 & 2 \end{pmatrix}$

Solution

Given $A = \begin{pmatrix} 2 & 3 & 1 \\ -3 & 2 & -4 \\ 4 & 6 & 2 \end{pmatrix}$

$$|A| = 2 \begin{vmatrix} -4 & -3 \\ 6 & 2 \end{vmatrix} - 3 \begin{vmatrix} -3 & -4 \\ 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} -3 & 2 \\ 4 & 6 \end{vmatrix}$$

$$|A| = 2(4 + 24) - 3(-6 + 16) + 1(-18 - 8)$$

$$|A| = 56 - 30 - 26 = 0, \text{ Matrix } A \text{ is a singular matrix}$$

Inverse of a matrix

Minors and cofactors of a 3 by 3 matrices: Minor is a value of a determinant obtained by deleting the row and column containing the element. Cofactor is the value of the minor with its respective matrix direction sign.

Adjoint matrix is a transpose of the matrix of cofactor

Inverse of matrix is adjoint over the determinant of the matrix. Inverse of matrix A is denoted as A^{-1} .

Mathematically, inverse of the matrix $A^{-1} = \frac{Adj(A)}{\det(A)}$, provided $\det(A) \neq 0$.

Any square matrix with none zero determinant has an inverse.

Example 75

Find inverse of matrix B if $B = \begin{pmatrix} 3 & -1 & 0 \\ 2 & -4 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Solution

Step 1: Find the determinant $B = \begin{pmatrix} 3 & -1 & 0 \\ 2 & -4 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$$|B| = 3 \begin{vmatrix} -4 & 1 \\ 1 & 3 \end{vmatrix} - (-1) \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & -4 \\ 0 & 1 \end{vmatrix}$$

$$|B| = 3(-12-1) + 1(6-0) + 0$$

Determinant $|B| = -39 + 6 = -33$

The determinant is -33

Step 2: Find the cofactors matrix. A 3 by 3 matrix has 9 cofactors. Let the

cofactor matrix of matrix $B = \begin{pmatrix} 3 & -1 & 0 \\ 2 & -4 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ be $B^C = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{pmatrix}$

When finding the cofactors, we use the matrix direction signs but we exclude the place element.

$$A_1 = + \begin{vmatrix} -4 & 1 \\ 1 & 3 \end{vmatrix} = -13, \quad A_2 = - \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = -6, \quad A_3 = + \begin{vmatrix} 2 & -4 \\ 0 & 1 \end{vmatrix} = 2,$$

$$A_4 = - \begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} = 3, \quad A_5 = + \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 9, \quad A_6 = - \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = -3,$$

$$A_7 = + \begin{vmatrix} -1 & 0 \\ -4 & 1 \end{vmatrix} = -1, \quad A_8 = - \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = -3, \quad A_9 = + \begin{vmatrix} 3 & -1 \\ 2 & -4 \end{vmatrix} = -10 \text{ then}$$

$$B^c = \begin{pmatrix} -13 & -6 & 2 \\ 3 & 9 & -3 \\ -1 & -3 & -10 \end{pmatrix}$$

Step 3: Find the adjoint matrix, the transpose of matrix of cofactors is an adjoint. That is $(B^c)^T = \text{adj}(A)$

$$(B^c)^T = \text{adj}(A) = \begin{pmatrix} -13 & 3 & -1 \\ -6 & 9 & -3 \\ 2 & -3 & -10 \end{pmatrix}$$

Step 4: Find the inverse of the matrix by taking adjoint over the determinant

$$B^{-1} = -\frac{1}{33} \begin{pmatrix} -13 & 3 & -1 \\ -6 & 9 & -3 \\ 2 & -3 & -10 \end{pmatrix}$$

Therefore the determinant is $B^{-1} = \begin{pmatrix} \frac{13}{33} & -\frac{1}{11} & \frac{1}{33} \\ \frac{2}{11} & -\frac{3}{11} & \frac{1}{11} \\ -\frac{2}{33} & \frac{1}{11} & \frac{10}{33} \end{pmatrix}$

Questions

(a) Use $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 4 \\ -6 & 0 \end{pmatrix}$ to show that $(AB)^{-1} = B^{-1}A^{-1}$

(b) If $X = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 3 \\ 0 & 1 & -2 \end{pmatrix}$ show that $X^{-1}Y^{-1} = (YX)^{-1}$

EXERCISE 13: ALGEBRA

1. If the $Q = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ a & b & c \end{pmatrix}$ is a singular matrix, show that $c = 4b - 5a$
2. If the determinant of matrix $P = \begin{pmatrix} 2 & 3 & -1 \\ 0 & -3 & 2 \\ x & y & -2 \end{pmatrix}$ is 12, show that $y = \frac{3}{4}x$
3. Given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} 1 & a \\ 3 & b \end{pmatrix}$ if $|A| = |B|$ and $a = b = c$ show that $d + 2 = a = b = c$
4. Find the inverse of $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 2 & 3 & 0 \end{pmatrix}$
5. If $X = \begin{pmatrix} 1 & 3 \\ 2 & a \end{pmatrix}$ is the inverse of $Y = \begin{pmatrix} 7 & -3 \\ b & 1 \end{pmatrix}$ find the values of a and b .
6. If $B = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$, Show that $B^{-1} = B^T$
7. Given $A = \begin{pmatrix} 3 & 1 \\ x & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 2 \\ 4 & y \end{pmatrix}$ find the values of x and y given that $AB = BA$
8. Given the matrix $M = \begin{pmatrix} 2 & 1 & -3 \\ 3 & 2 & -3 \\ -2 & 1 & 3 \end{pmatrix}$ find
 - (a) The order of M
 - (b) The transpose of M
 - (c) The determinant of M
 - (d) The adjoint of M
 - (e) The inverse of M
9. If $\begin{pmatrix} x+1 & 2 \\ 4 & x-1 \end{pmatrix}$ is a singular matrix find the possible values of x .

10. The matrices $A = \begin{pmatrix} 5 & 2 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are such that $AB = A + B$.

Find the value of a, b, c and d .

11. If the matrix $A = \begin{pmatrix} 1 & -2 & -1 \\ 7 & 0 & -1 \\ 1 & -2 & 3 \end{pmatrix}$ Verify that $\det A = \det A^T$

12. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}$

13. Find the additive inverse of the matrix B if $B = \begin{pmatrix} -5 & 6 \\ 0 & -9 \end{pmatrix}$

14. If $A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 4 & 0 \\ 3 & -2 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 8 & -7 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 3 \\ 3 & 3 & 0 \\ 1 & 1 & 4 \end{pmatrix}$ find

(a) $A + 3B - 2C$

(b) $AB + C - 3I$ Where I is the identity matrix of order 3 by 3

(c) Is $A(B+C)$ and $AB+AC$ equals?

15. Show that if $A = \begin{pmatrix} 5 & 7 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -5 \\ 9 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$, then

$$|A+B-2C| = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix}$$

6.11.1 Solution of simultaneous equations

Simultaneous equations can be solved in several methods; matrix (inverse) method and Cramer's (determinant) method are also used to solve a simultaneous equations.

(a) Matrix method / inverse method

Consider the linear simultaneous of three unknown variables

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases} \text{ where all coefficients are constants.}$$

The augmented matrix is the matrix of the coefficients

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ then from the result above,

$A \times B = C$ make B the subject to get, $B = A^{-1}C$ where B is a matrix of unknowns x, y and z .

Example 76

Solve by matrix method $\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases}$

Solution

Given the equations $\begin{cases} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{cases}$

The matrices $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ -2 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} 6 \\ 14 \\ -2 \end{pmatrix}$

Using the rule $B = A^{-1}C$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix} \text{ thus } |A| = 1 \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$$

$$|A| = 50$$

Cofactors, let the matrix of cofactors be $A^c = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ then

$$\begin{aligned} a_1 &= + \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} = 1, & a_2 &= - \begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix} = 8, & a_3 &= + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 11 \\ a_4 &= - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5, & a_5 &= + \begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} = -10, & a_6 &= - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 5 \\ a_7 &= + \begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} = 13, & a_8 &= - \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 4, & a_9 &= + \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -7. \end{aligned}$$

The matrix of cofactor is $A^c = \begin{pmatrix} 1 & 8 & 11 \\ 5 & -10 & 5 \\ 13 & 4 & 7 \end{pmatrix}$ and

$$A^{-1} = \frac{1}{50} \begin{pmatrix} 1 & 5 & 13 \\ 8 & -10 & 4 \\ 11 & 5 & -7 \end{pmatrix}$$

$$B = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 1 & 5 & 13 \\ 8 & -10 & 4 \\ 11 & 5 & -7 \end{pmatrix} \begin{pmatrix} 6 \\ 14 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{50} \begin{pmatrix} 50 \\ -100 \\ 150 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Therefore the solution are $x=1, y=-2, z=3$.

(b) Cramer's rule / determinant method

Cramer's method was introduced by the first time by Gabriel Cramer 1704 – 1752. The method used to solve the system of linear equations with as many equations as unknown variables, which is valid whenever the system has a unique solution. Solving using Cramer's rule, the solution is expressed in determinant form.

Example: Suppose the system of linear equations
$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Step 1: Form the matrix of coefficients (augmented matrix)

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Step 2: Compute the value of unknowns

$$x = \frac{\Delta_x}{\Delta_0} \Rightarrow x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\Delta_y}{\Delta_0} \Rightarrow y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = \frac{\Delta_z}{\Delta_0} \Rightarrow z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \text{ where } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

Example 77

Solve by Cramer's rule
$$\begin{cases} x + 3y + z = 5 \\ 3x - 2y + 4z = -8 \\ x + 2z = -2 \end{cases}$$

Solution

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & -2 & 4 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \\ -2 \end{pmatrix}$$

$$x = \frac{\begin{vmatrix} 5 & 3 & 1 \\ -8 & -2 & 4 \\ -2 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 1 \\ 3 & -2 & 4 \\ 1 & 0 & -2 \end{vmatrix}}} = \frac{5 \begin{vmatrix} -2 & 4 \\ 0 & 2 \end{vmatrix} - 3 \begin{vmatrix} -8 & 4 \\ -2 & 2 \end{vmatrix} + 1 \begin{vmatrix} -8 & -2 \\ -2 & 0 \end{vmatrix}}{1 \begin{vmatrix} -2 & 4 \\ 0 & -2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix}}} = \frac{0}{-8} = 0$$

$$y = \frac{\begin{vmatrix} 1 & 5 & 1 \\ 3 & -8 & 4 \\ 1 & -2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 1 \\ 3 & -2 & 4 \\ 1 & 0 & -2 \end{vmatrix}}} = \frac{1 \begin{vmatrix} -8 & 4 \\ -2 & 2 \end{vmatrix} - 5 \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & -8 \\ 1 & -2 \end{vmatrix}}{1 \begin{vmatrix} -2 & 4 \\ 0 & -2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix}}} = \frac{-16}{-8} = 2$$

$$z = \frac{\begin{vmatrix} 1 & 3 & 5 \\ 3 & -2 & -8 \\ 1 & 0 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 1 \\ 3 & -2 & 4 \\ 1 & 0 & -2 \end{vmatrix}}} = \frac{1 \begin{vmatrix} -2 & -8 \\ 0 & -2 \end{vmatrix} - 3 \begin{vmatrix} 3 & -8 \\ 1 & -4 \end{vmatrix} + 5 \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix}}{1 \begin{vmatrix} -2 & 4 \\ 0 & -2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} + 1 \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix}}} = \frac{8}{-8} = -1$$

Therefore $x = 0$, $y = 2$ and $z = -1$

Types of solution of linear simultaneous equations

Note that:

1. If the number of equations is greater than or equal to the number of variables (unknowns) in a linear system of equations, then one of the following is true:
 - (a) The system has no solution.
 - (b) The system has exactly one solution.
 - (c) The system has infinitely many solutions.
2. If there are fewer equations than variables in a linear system, then either the system has no solution or it has infinitely many solutions.
3. If there is a row in an augmented matrix containing all zeros to the left of the vertical line and a nonzero entry to the right of the line, then the corresponding system of equations has no solution.

(a) Unique or exactly one solution (Consistent solution)

Consider the equations
$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

The equations can be written as
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix},$$

Then the equation has unique solution if
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

Example 78

Determine whether the equations has unique solution or not

$$\begin{cases} 3x + y - z = 4 \\ x + y - z = 2 \\ x + 3y + 2z = 9 \end{cases} \quad \text{if so determine the solutions.}$$

Solution

Given
$$\begin{cases} 3x + y - z = 4 \\ x + y - z = 2 \\ x + 3y + 2z = 9 \end{cases}$$

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 9 \end{pmatrix}$$

$$\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 10 \neq 0$$

Since the determinant of matrix, $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 3 & 2 \end{pmatrix}$ is not zero the equations has

a unique solutions.

The solutions by Cramer's rule

$$x = \frac{\Delta_x}{\Delta_0}, \quad y = \frac{\Delta_y}{\Delta_0} \quad \text{and} \quad z = \frac{\Delta_z}{\Delta_0}$$

$$x = \frac{\begin{vmatrix} 4 & 1 & -1 \\ 2 & 1 & -1 \\ 9 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix}} = \frac{10}{10} = 1, \quad y = \frac{\begin{vmatrix} 3 & 4 & -1 \\ 1 & 2 & -1 \\ 1 & 9 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix}} = \frac{20}{10} = 2 \text{ and}$$

$$z = \frac{\begin{vmatrix} 3 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & 3 & 9 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 3 & 2 \end{vmatrix}} = \frac{10}{10} = 1$$

The solutions are $x = 1$, $y = 2$ and $z = 1$.

Example 79

For what value of p is the system of equations $\begin{cases} x - 4y + 2z = 2 \\ 4x + 3y + 2z = 1 \\ x + py = -3 \end{cases}$ have no unique solution?

Solution

Given $\begin{cases} x - 4y + 2z = 2 \\ 4x + 3y + 2z = 1 \\ x + py = -3 \end{cases}$

$$\begin{pmatrix} 1 & -4 & 2 \\ 4 & 3 & 2 \\ 1 & p & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

For the system to have no unique solution $\begin{vmatrix} 1 & -4 & 2 \\ 4 & 3 & 2 \\ 1 & p & 0 \end{vmatrix} = 0$

$$1 \begin{vmatrix} 3 & 2 \\ p & 0 \end{vmatrix} + 4 \begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix} + 2 \begin{vmatrix} 4 & 3 \\ 1 & p \end{vmatrix} = 0 \Rightarrow -2p - 8 + 8p - 6 = 0$$

$$6p - 14 = 0 \Rightarrow p = \frac{7}{3} \text{ for the system of equations to have no unique solution,}$$

$$p = \frac{7}{3}$$

(b) Infinity many solution (Consistence with many solutions)

Infinity many solution happens when all lines coincides. Consider the lines

$$\begin{cases} x + 3y = 4 \\ 2x + 6y = 8 \end{cases}$$

The lines are coincides, and have so many common points which are called many solution, the possible set of solutions of the equations are as shown in this table below

x	1	4	0	3	22	...
y	1	0	4/3	1/3	6	...

There are many more sets of x and y which satisfy the two equations. Using matrix, to determine whether the equations given has many solution

or not, consider again the same equations above
$$\begin{cases} x + 3y = 4 \\ 2x + 6y = 8 \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$$\Delta_x = \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 24 - 24 = 0, \text{ because determinant obtained by replacing the}$$

column of x has also zero determinant, then the equations has infinity many solutions.

$$\text{Because, } x = \frac{\Delta_x}{\Delta_0} = \frac{\begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}} = \frac{0}{0} \text{ and } y = \frac{\Delta_y}{\Delta_0} = \frac{\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}} = \frac{0}{0} \text{ leads to}$$

undetermined condition, which conclude that the equations has infinity many solution.

The solutions of the equations can be obtained by letting $x = t$, so that

$$y = \frac{1}{3}(4 - t) \text{ where } t \text{ is any real number.}$$

$$\text{Now consider the equations } \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

$$\text{The equations can be written as } \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\text{Let } A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, B = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } C = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Note that: If, $\text{Adj}(A) \times C = 0$, where 0 is zero matrix the equations are consistent with infinity many solutions and if $\text{Adj}(A) \times C \neq 0$ then the system has no solution.

Example 80

Show that $\begin{cases} 3x + y = 5 \\ 6x + 2y = 10 \end{cases}$ has infinity many solutions

Solution

Given $\begin{cases} 3x + y = 5 \\ 6x + 2y = 10 \end{cases} \Rightarrow \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$

Let $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$, $B = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$

Many solution test, $\text{Adj}(A) \times C = 0$ (zero matrix)

$$\text{Adj}(A) = \begin{pmatrix} -2 & 1 \\ 6 & -3 \end{pmatrix}$$

$$\text{Adj}(A) \times C = \begin{pmatrix} -2 & 1 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore, the system has infinity many solutions.

From $\begin{cases} 3x + y = 5 \\ 6x + 2y = 10 \end{cases}$

Then $3x + y = 5$ therefore, $y = 5 - 3x$, if $x = t$ then $y = 5 - 3t$.

Example 81

Determine if the equations $\begin{cases} x + 2y + z = 1 \\ 2x + 4y + 2z = 2 \\ -x - 2y - z = -1 \end{cases}$ has unique or many solutions.

Solution

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

For unique solution $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -2 & -1 \end{vmatrix} \neq 0$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -2 & -1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} = 0 \text{ since determinant is}$$

zero, the equations has no unique solutions, it can either have many solution or no solution.

To test for many solution use the undetermined condition of $0/0$,

$$x = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -2 & -1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -2 & -1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ -1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix}} = \frac{0}{0}.$$

Because the system lead to undetermined condition, then the equations has infinity many solutions.

(c) No solution

The systems of equations $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ are said to have no solution

if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ and $Adj(A) \times C \neq 0$

Example 82

Show that the system of equations $\begin{cases} 2x - 3y = 7 \\ 4x - 6y = 10 \end{cases}$ has no solution.

Solution

$$\begin{cases} 2x - 3y = 7 \\ 4x - 6y = 10 \end{cases} \text{ thus } \begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

$$\text{Let } A = \begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix}, B = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } C = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

$$\text{Adj}(A) = \begin{pmatrix} -6 & 3 \\ -4 & 2 \end{pmatrix}$$

$$\text{Adj}(A) \times C = \begin{pmatrix} -6 & 3 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix} = \begin{pmatrix} -12 \\ -8 \end{pmatrix} \neq 0$$

Therefore, the equations has no solution (inconsistent).

Example 83

$$\text{Show that } \begin{cases} x + 2y + z = 4 \\ 3x + 4y + 2z = 7 \\ x - 2y - z = 5 \end{cases} \text{ has no solution.}$$

Solution

$$\text{Given } \begin{cases} x + 2y + z = 4 \\ 3x + 4y + 2z = 7 \\ x - 2y - z = 5 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix}$$

$$\text{Find the adjoint of } \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & -2 & -1 \end{pmatrix}$$

Let the cofactors matrix be $A^c = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$

$$a_1 = + \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} = 0, \quad a_2 = - \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = 5, \quad a_3 = + \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = -10,$$

$$a_4 = - \begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} = 0, \quad a_5 = + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2, \quad a_6 = - \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 4,$$

$$a_7 = + \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0, \quad a_8 = - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = 1, \quad a_9 = + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$A^c = \begin{pmatrix} 0 & 5 & -10 \\ 0 & -2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\text{Adj}(A) = \begin{pmatrix} 0 & 0 & 0 \\ 5 & -2 & 1 \\ -10 & 4 & -2 \end{pmatrix}$$

$$\text{Adj}(A) \times C = \begin{pmatrix} 0 & 0 & 0 \\ 5 & -2 & 1 \\ -10 & 4 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 11 \\ -22 \end{pmatrix}$$

Therefore, $\text{Adj}(A) \times C \neq 0$, hence the equations has no solution

Example 84

Find the relation connecting p and k so that the system of simultaneous

$$\text{equations } \begin{cases} x + y - pz = 4 \\ 3x + 2y + z = 3 \\ 5x + 4y + 6z = k \end{cases} \text{ has no solution}$$

Solution

$$\begin{cases} x+y-pz=4 \\ 3x+2y+z=3 \\ 5x+4y+6z=k \end{cases} \quad \text{thus} \quad \begin{pmatrix} 1 & 1 & -p \\ 3 & 2 & 1 \\ 5 & 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ k \end{pmatrix}$$

$$\Delta_0 = \begin{vmatrix} 1 & 1 & -p \\ 3 & 2 & 1 \\ 5 & 4 & 6 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} 4 & 1 & -p \\ 3 & 2 & 1 \\ k & 4 & 6 \end{vmatrix} = 4 \begin{vmatrix} 2 & 1 \\ 4 & 6 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ k & 6 \end{vmatrix} - p \begin{vmatrix} 3 & 2 \\ k & 4 \end{vmatrix} \neq 0$$

$$32 - 18 + k - 12p + 2pk \neq 0$$

The required relation is $12p - 2pk - k \neq 14$

Example 85

Solve for x , y and z
$$\begin{cases} x+2y+z=7 \\ 3x+y=-2 \\ 5x+5y+2z=12 \end{cases}$$

Solution

Given
$$\begin{cases} x+2y+z=7 \\ 3x+y=-2 \\ 5x+5y+2z=12 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 5 & 5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 12 \end{pmatrix}$$

$$\Delta_0 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 5 & 5 & 2 \end{vmatrix} = -3 \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} = 3 - 3 = 0$$

The original determinant is zero, this means equations has no unique solution, can either have infinity many solutions, $Adj(A) \times C = 0$ or has no solution, $Adj(A) \times C \neq 0$

Let $A^C = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$ be the cofactors matrix,

From $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 5 & 5 & 2 \end{pmatrix}$

$$a_1 = + \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = 2, \quad a_2 = - \begin{vmatrix} 3 & 0 \\ 5 & 2 \end{vmatrix} = -6, \quad a_3 = + \begin{vmatrix} 3 & 1 \\ 5 & 5 \end{vmatrix} = 10, \quad a_4 = - \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = 1,$$

$$a_5 = + \begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} = -3, \quad a_6 = - \begin{vmatrix} 1 & 2 \\ 5 & 5 \end{vmatrix} = 5, \quad a_7 = + \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad a_8 = - \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = 3,$$

$$a_9 = + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5 \text{ then cofactor matrix become, } A^C = \begin{pmatrix} 2 & -6 & 10 \\ 1 & -3 & 5 \\ -1 & 3 & -5 \end{pmatrix} \text{ the}$$

$$\text{adjoint of A is } Adj(A) = \begin{pmatrix} 2 & 1 & -1 \\ -6 & -3 & 3 \\ 10 & 5 & -5 \end{pmatrix}$$

$$Adj(A) \times C = \begin{pmatrix} 2 & 1 & -1 \\ -6 & -3 & 3 \\ 10 & 5 & -5 \end{pmatrix} \begin{pmatrix} 7 \\ -2 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Because $Adj(A) \times C = 0$, the system is consistent with infinity many solutions.

$$\text{These solutions can be obtained by letting } x = t, \begin{cases} x + 2y + z = 7 \\ 3x + y = -2 \\ 5x + 5y + 2z = 12 \end{cases}$$

From $x + 2y + z = 7 \Rightarrow z = 7 - x - 2y$ because $x = t$ it become $z = 7 - t - 2y$

From $3x + y = -2$ then $y = -2 - 3x \Rightarrow y = -2 - 3t$

Form the two equations above, $z = 7 - t - 2y$ and $y = -2 - 3t$ substituting y in an equation containing z to get z . $z = 7 - t - 2(-2 - 3t)$, $z = 11 + 5t$

Therefore the solutions are $x = t$, $y = -2 - 3t$ and $z = 11 + 5t$

(d) Trivial and non-trivial solution

Consider the system of equations
$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

If $d_1 = d_2 = d_3 = 0$ and $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ the equations are called trivial

equations, where $x = y = z = 0$ are called trivial solutions.

Non-trivial solution is a non-zero solution of the equations.

Example 86

Find a non-trivial solution of the following equations
$$\begin{cases} x + y - z = 0 \\ 5x - y - z = 0 \\ 4x + y - 2z = 0 \end{cases}$$

Solution

Given
$$\begin{cases} x + y - z = 0 \\ 5x - y - z = 0 \\ 4x + y - 2z = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 5 & -1 & -1 \\ 4 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Test: } \begin{vmatrix} 1 & 1 & -1 \\ 5 & -1 & -1 \\ 4 & 1 & -2 \end{vmatrix} = 1 \begin{vmatrix} -1 & -1 \\ 1 & -2 \end{vmatrix} - 1 \begin{vmatrix} 5 & -1 \\ 4 & -2 \end{vmatrix} - 1 \begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix} = 0$$

The equations has trivial solutions $x = y = z = 0$

To get a non-trivial solution, let $x = t$

$$\text{Add the first two equations } + \begin{cases} x + y - z = 0 \\ 5x - y - z = 0 \end{cases}$$

$$6x = 2z \text{ which leads to } z = 3x \Rightarrow z = 3t$$

Choose any equation and make y the subject, here equation 3,
 $4x + y - 2z = 0$

$$y = 2z - 4x, \text{ given, } x = t \text{ and } z = 3t \Rightarrow y = 2(3t) - 4t \Rightarrow y = 2t$$

Therefore, the non-trivial solutions are $x = t$, $y = 2t$ and $z = 3t$. Where t is a real number.

Questions

1. When visiting an insane asylum, I asked two inmates to give me their ages. They did so, and then, to test their arithmetical powers, I asked them to add the two ages together. One gave me 44 as the answer, and the other gave 1,280. I immediately saw that the first had subtracted one age from the other, while the second person had multiplied them together. What were their ages?
2. A man and his wife had three children, John, Ben, and Mary, and the difference between their parents' ages was the same as between John and Ben and between Ben and Mary. The ages of John and Ben, multiplied together, equalled the age of the father, and the ages of Ben and Mary multiplied together equalled the age of the mother. The combined ages of the family amounted to ninety years. What was the age of each person?

EXERCISE 14: ALGEBRA

1. Use the matrix method solve for x , y and z from
$$\begin{cases} 2x + y + z = 180 \\ x + 3y + 2z = 300 \\ 2x + y + 2z = 240 \end{cases}$$

Verify the solutions by Cramer's rule.

2. Solve the system of simultaneous equations by either method

$$\begin{cases} 2x_1 + 4x_2 + 6x_3 = 22 \\ 3x_1 + 8x_2 + 5x_3 = 27 \\ -x_1 + x_2 + 2x_3 = 2 \end{cases}$$

3. Solve for x , y and z by inverse method
$$\begin{cases} 3x - 2y + 8z = 9 \\ -2x + 2y + z = 3 \\ x + 2y - 3z = 8 \end{cases}$$

4. Solve the system of simultaneous equations
$$\begin{cases} 2y + 3z = 7 \\ 3x + 6y - 12z = -3 \\ 5x - 2y + 2z = -7 \end{cases}$$

5. Solve by Cramer's method
$$\begin{cases} x + y - z = -3 \\ 3x + y + 2z = 9 \\ 4x - 3y + z = 1 \end{cases}$$

Examine the consistence of the following simultaneous equations

$$6. \begin{cases} 5x - y + 4z = 5 \\ 2x + 3y + 5z = 2 \\ 5x - 2y + 6z = -1 \end{cases} \quad 7. \begin{cases} 3x_1 - x_2 - 2x_3 = 2 \\ 2x_2 - x_3 = -1 \\ 3x_1 - 5x_2 = 3 \end{cases}$$

$$8. \begin{cases} x + 3y + 5z = 10 \\ 3x + 2y - z = 7 \\ 4x + 5y + 4z = 17 \end{cases} \quad 9. \begin{cases} 4x + 8y + 2z = 8 \\ 5x - y - 3z = 4 \\ 3x + 5y + z = 7 \end{cases}$$

$$10. \begin{cases} 7x + 3y + 4z = 30 \\ x + 8y - 4z = 80 \\ 6x - 5y + 8z = 25 \end{cases} \quad 11. \begin{cases} x + 2y + 3z = 4 \\ 4x + 5y + 6z = 10 \\ 5x + 7y + 9z = 14 \end{cases}$$

$$12. \begin{cases} 8x + 4y + 8z = 2 \\ -12x + 5y - z = -3 \\ 4x + 3y + 5z = 1 \end{cases} \quad 13. \begin{cases} x + 2y - 3z = -2 \\ 3x - y - 2z = 1 \\ 2x + 3y - 5z = -3 \end{cases}$$

14. Find the value of k for which the system of equations

$$\begin{cases} x + y + 5z = 11 \\ 2x + 2y - z = 3 \\ kx + y - 3z = 1 \end{cases} \text{ have unique solution}$$

Find a non-trivial solution of equations

$$15. \begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 5x + 7y + 9z = 0 \end{cases}$$

$$16. \begin{cases} 2x - y + 4z = 0 \\ 3x + 4y + 12z = 0 \\ 7x - 9y + 8z = 0 \end{cases}$$

$$17. \begin{cases} 4x - 2y + 2z = 0 \\ 3x + 4y + 7z = 0 \\ x + 5y + 6z = 0 \end{cases}$$

$$18. \begin{cases} 4x + 7y + 2z = 0 \\ 3y - 2z = 0 \\ 4x + 7y + 2z = 0 \end{cases}$$

$$19. \begin{cases} 2x + 4y + 8z = 0 \\ -3x + 5y - z = 0 \\ x + 3y + 5z = 0 \end{cases}$$

$$20. \begin{cases} 7x + 3y + 4z = 0 \\ x + 8y - 4z = 0 \\ 6x - 5y + 8z = 0 \end{cases}$$

$$21. \text{ Show that } \begin{vmatrix} x & a & x-a \\ y & b & y-b \\ z & c & z-c \end{vmatrix} = 0$$

$$22. \text{ Prove that } \begin{vmatrix} a & a & a \\ a & b & b \\ a & b & c \end{vmatrix} = a(b-a)(c-b)$$

Question

A number is made up of two digits. The sum of its digits is 11. If the digits in the original number are interchanged, the original number is increased by 9. Find the number.

6.11.3 Properties of determinant

Property 1

If columns are changed to rows or rows to columns, the value of the determinant remain unchanged. (Determinant transpose).

Let matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant is $\det(A) = ad - bc$. Let A

transpose be $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ the determinant is $\det(A^T) = ad - bc$ shown

For a 3 by 3 matrices, let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Determinant of A is $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The determinant of A using the first row is

$$\det(A) = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\det(A) = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1$$

Let the transpose of A be $A^T = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Determinant of transpose of A is $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Determinant of transpose of A using the first column is

$$\det(A^T) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

$\det(A^T) = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1$ hence proved.

Property 2

If two rows or columns has identical elements, the value of the determinant is zero.

Proof:

$$1. \text{ Let the determinant be } |A| = \begin{vmatrix} a & a \\ b & b \end{vmatrix} = ab - ab = 0$$

$$2. \text{ Let the determinant be } |A| = \begin{vmatrix} a & a & a \\ a & a & a \\ b & b & b \end{vmatrix} = a \begin{vmatrix} a & a \\ b & b \end{vmatrix} - a \begin{vmatrix} a & a \\ b & b \end{vmatrix} + a \begin{vmatrix} a & a \\ b & b \end{vmatrix} = 0$$

Example 87

Find the determinant of $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 5 & 1 & 6 \end{pmatrix}$

Solution

$$|A| = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 5 & 1 & 6 \end{vmatrix} \text{ thus } |A| = 2 \begin{vmatrix} 2 & 2 \\ 1 & 6 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 5 & 6 \end{vmatrix} + 2 \begin{vmatrix} 2 & 2 \\ 1 & 6 \end{vmatrix}$$

$$|A| = 2(12 - 2) - 2(12 - 10) + 2(2 - 10)$$

$$|A| = 20 - 4 - 16 = 0$$

Property 3

If a scalar number is multiplied to the determinant, that scalar is multiplied to only one row or one column of a determinant.

Thus,

$$\text{Let } k|A| = k \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} ka_1 & a_2 & a_3 \\ kb_1 & b_2 & b_3 \\ kc_1 & c_2 & c_3 \end{vmatrix}$$

Vice-versa is true.

Property 4

Interchanging rows or columns of the determinant changes the sign of the determinant.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proof

$$\text{Let } \Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\Delta_1 = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\text{Let } \Delta_2 = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\Delta_2 = b_1(a_2c_3 - a_3c_2) - b_2(a_1c_3 - a_3c_1) + b_3(a_1c_2 - a_2c_1)$$

$$\Delta_2 = -a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) - a_3(b_1c_2 - b_2c_1)$$

Thus $\Delta_1 = -\Delta_2$

Property 5

If each element of one row or column of the determinant is multiplied by constant, and the results is added or subtracted to the other row or column, the determinant is unchanged.

$$\text{That is } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + ka_1 & b_2 + ka_2 & b_3 + ka_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ where } k \text{ is any real}$$

number.

Proof

$$\text{Let } \Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\Delta_1 = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\text{Let } \Delta_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + ka_1 & b_2 + ka_2 & b_3 + ka_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\Delta_2 = a_1 \begin{vmatrix} b_2 + ka_2 & b_3 + ka_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 + ka_1 & b_3 + ka_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 + ka_1 & b_2 + ka_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\Delta_2 = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\text{Hence } \Delta_1 = \Delta_2$$

Property 6

If the elements of one row or column of the determinant are multiples of the other row or column, then the determinant is zero.

$$\text{That is } \begin{vmatrix} a_1 & a_2 & a_3 \\ 2a_1 & 2a_2 & 2a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Proof

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ 2a_1 & 2a_2 & 2a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} 2a_2 & 2a_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} 2a_1 & 2a_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} 2a_1 & 2a_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\Delta = a_1(2a_2c_3 - 2a_3c_2) - a_2(2a_1c_3 - 2a_3c_1) + a_3(2a_1c_2 - 2a_2c_1)$$

$$\Delta = 2a_1a_2c_3 - 2a_1a_3c_2 - 2a_1a_2c_3 + 2a_2a_3c_1 + 2a_1a_3c_2 - 2a_2a_3c_1 = 0$$

Thus, $\Delta = 0$

Property 7

Product of two determinants is equals to the product of individual determinant.

$$\text{If } A = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_3 \end{vmatrix} \text{ and } B = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \text{ then } \det(AB) = \det(A) \times \det(B)$$

Example 88

Using the properties of determinant, show that $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$

Solution

$$\begin{aligned}
 & \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} \Rightarrow C_1 = C_1 + C_2 + C_3 \\
 & = \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} = (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \\
 & \quad (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \\
 & \quad (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix} \Rightarrow \begin{cases} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{cases} \\
 & = (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 1-1 & 1-x & x-x^2 \\ 1-1 & x^2-x & 1-x^2 \end{vmatrix} \\
 & \quad (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x(1-x) \\ 0 & -x(1-x) & (1-x)(1+x) \end{vmatrix} \\
 & = (1+x+x^2)(1-x)(1-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & 1+x \end{vmatrix}
 \end{aligned}$$

$$\begin{aligned} (1+x+x^2)(1-x)^2 \begin{vmatrix} 1 & x \\ -x & 1+x \end{vmatrix} &= (1+x+x^2)(1-x)^2(1+x+x^2) \\ &= (1+x+x^2)(1-x)^2 = (1-x^3)^2 \text{ hence shown.} \end{aligned}$$

Example 89

Use the determinant properties, show that $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & y^2z & 0 \end{vmatrix} = 2(xyz)^3$

Solution

$$\begin{aligned} \text{Given } & \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & y^2z & 0 \end{vmatrix} \\ & \begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & y^2z & 0 \end{vmatrix} = x^2y^2z^2 \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix} \\ & \Rightarrow C_2 = C_2 - C_3 = (xyz)^2 \begin{vmatrix} 0 & x-x & x \\ y & 0-y & y \\ z & z-0 & z \end{vmatrix} \\ & = (xyz)^2 \begin{vmatrix} 0 & 0 & x \\ y & -y & y \\ z & z & 0 \end{vmatrix} = (xyz)^2 (x(yz + yz)) = 2(xyz)^3 \end{aligned}$$

Hence shown.

Example 90

$$\text{Prove that } \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$$

Solution

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \Rightarrow \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$= xyz \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & (y-x)(y+x) & (z-x)(z+x) \end{vmatrix}$$

$$xyz(y-x)(z-x) \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix} = xyz(y-x)(z-x)(z-y) \text{ hence shown}$$

EXERCISE 15: ALGEBRA

1. Use any 3 by 3 determinant; prove that when columns or rows are interchanged the determinant changes the sign.

2. Use determinant properties factorize $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

3. Show that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = w^2 + w + 1 = 0$

4. Prove that $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)(x-z)^2$

5. Factorize $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

6. Factorize $\begin{vmatrix} a & bc & a^2 \\ b & ac & b^2 \\ c & ab & c^2 \end{vmatrix}$

7. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

8. Factorize $\begin{vmatrix} x & x^3 & x^4 \\ y & y^3 & y^4 \\ z & z^3 & z^4 \end{vmatrix}$

9. Show that
$$\begin{vmatrix} y & x & x+y \\ x+y & y & x \\ x & x+y & y \end{vmatrix} = 2(x^3 - y^3)$$

10. Show that
$$\begin{vmatrix} (y+z)^2 & x^2 & x^2 \\ y^2 & (x+z)^2 & y^2 \\ z^2 & z^2 & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3$$

Pascal's triangle

$$(1+x)^0 = 1$$

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

.....

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + x^n \text{ which in short form can be written as } (1+x)^n = {}^nC_0x^0 + {}^nC_1x^1 + {}^nC_2x^2 + \dots + {}^nC_rx^r + \dots + {}^nC_nx^n \text{ where } {}^nC_r = \frac{n!}{(n-r)!r!}$$

So when n is a positive integer the expansion gives a finite series called binomial expansion.

When $n = 4$

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 \quad \text{simplified series become} \\ (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4 \quad \text{the series terminate when } x \text{ reaches the power of 4.}$$

Note that, in the expansion $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$, $n = 4$ and the number of terms is 5. So that when n is a positive integer the expansion will have $(n+1)$ terms.

When n is a negative integer is the numerator of the coefficient will never contain a factor zero. In that case $(n-r+1)$ will never be zero, therefore the expansion does not terminate and the results is an infinity series. These results apply when n is a fraction.

Generally,

$$(a+b)^n = a^n + \frac{na^{n-1}}{1!}b + \frac{n(n-1)a^{n-2}}{2!}b^2 + \frac{n(n-1)(n-2)a^{n-3}}{3!}b^3 + \dots + b^n$$

Example 91

Expand $(2+x)^5$

Solution

Using binomial theorem

$$(2+x)^5 = {}^5C_0(2^5)x^0 + {}^5C_1(2^4)x^1 + {}^5C_2(2^3)x^2 + {}^5C_3(2^2)x^3 + {}^5C_4(2^1)x^4 + {}^5C_5(2^0)x^5 \\ (2+x)^5 = 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$$

Example 92

Expand $(1-2x)^{-4}$ up to the term containing x^4

Solution

$$\text{Using } (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots + \infty$$

$$\begin{aligned}(1-2x)^{-4} &= 1 + (-4)(-2x) + \frac{-4(-4-1)(-2x)^2}{2!} + \frac{-4(-4-1)(-4-2)(-2x)^3}{3!} + \\ &\frac{-4(-4-1)(-4-2)(-4-3)(-2x)^4}{4!} + \dots\end{aligned}$$

$$(1-2x)^{-4} = 1 + 8x + 40x^2 + 160x^3 + 560x^4$$

Note that if n is negative, combination formula cannot be used, because $(-n)!$ is not defined, likewise when n is fraction.

Example 93

Expand $(2+x)^{-1}$ up to the term contain x^4

Solution

To easy expansion, we re-write the expansion as

$$(2+x)^{-1} = \left(2\left(1+\frac{x}{2}\right)\right)^{-1} = 2^{-1}\left(1+\frac{x}{2}\right)^{-1} = \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$

$$\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{1}{2}\left(1 - \left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)x^3}{3!}\left(\frac{x}{2}\right)^3 + \dots\right)$$

$$\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{1}{2}\left(1 - \frac{x}{2} + \frac{2}{2}\left(\frac{x^2}{4}\right) - \frac{6}{6}\left(\frac{x^3}{8}\right) + \frac{24}{24}\left(\frac{x^4}{16}\right) + \dots\right)$$

$$\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} = (2+x)^{-1} = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \dots$$

Example 94

Expand $(x+y)^{10}$ in descending powers of x up to the term containing x^6

Solution

$$(x+y)^{10} = x^{10} + 10x^9y + \frac{10 \times 9x^8y^2}{2!} + \frac{10 \times 9 \times 8x^7y^3}{3!} + \frac{10 \times 9 \times 8 \times 7x^6y^4}{4!} + \dots$$

$$(x+y)^{10} = x^{10} + 10x^9y + 45x^8y^2 + 120x^7y^3 + 210x^6y^4 + \dots$$

Example 95

Use the binomial theorem to expand $(3+x)^7$ and use the results to estimate the value of $(3.045)^7$ correct to 7 significant figures.

Solution

$$(3+x)^7 = \left(3 \left(1 + \frac{x}{3} \right) \right)^7 = 3^7 \left(1 + \frac{x}{3} \right)^7$$

Expanding up to the term containing x^4 and ignore the higher terms,

$$3^7 \left(1 + \frac{x}{3} \right)^7 = 3^7 \left(1 + \frac{7x}{3} + \frac{7 \times 6}{2!} \left(\frac{x}{3} \right)^2 + \frac{7 \times 6 \times 5}{3!} \left(\frac{x}{3} \right)^3 + \frac{7 \times 6 \times 5 \times 4}{4!} \left(\frac{x}{3} \right)^4 + \dots \right)$$

$$3^7 \left(1 + \frac{x}{3} \right)^7 = 3^7 \left(1 + \frac{7x}{3} + \frac{7}{3}x^2 + \frac{35}{27}x^3 + \frac{35}{81}x^4 + \dots \right)$$

$$3^7 \left(1 + \frac{x}{3} \right)^7 = 2187 + 5103x + 5103x^2 + 2835x^3 + 945x^4 + \dots$$

$$\text{From } (3.045)^7 = (3 + 0.045)^7 = (3+x)^7$$

$$\text{Then } x = 0.045$$

$$3^7 \left(1 + \frac{x}{3} \right)^7 = 2187 + 5103x + 5103x^2 + 2835x^3 + 945x^4 + \dots$$

$$\text{Thus } (3.045)^7 \approx 2427.231$$

Example 96

Use binomial expansion find the value of $(2.007)^9$ correct to 4 decimal places

Solution

Let expand $(a+b)^n$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \dots$$

Ignoring the higher powers

$$\begin{aligned}(2.007)^9 &= (2+0.07)^9 = 2^9 + 9 \times 2^8 \times (0.007) + \frac{9 \times 8 \times 2^7 \times (0.007)^2}{2!} \\ &+ \frac{9 \times 8 \times 7 \times 2^6 \times (0.007)^3}{3!} + \dots\end{aligned}$$

$$(2.007)^2 = 512 + 16.128 + 0.225792 + 0.001843968 = 528.3556$$

Example 97

Use binomial theorem, expand $(2+x+x^2)^6$ as far as the term containing x^4

Solution

Given $(2+x+x^2)^6$

Let $b = (x+x^2)$

From $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \dots$

$$(2+b)^6 = 2^6 + 6 \times 2^5 \times b + \frac{6 \times 5 \times 2^4 \times b^2}{2!} + \frac{6 \times 5 \times 4 \times 2^3 \times b^3}{3!} + \dots$$

$$(2+b)^6 = 64 + 192b + 240b^2 + 160b^3 + \dots$$

$$(2+x+x^2)^6 = 64 + 192(x+x^2) + 240(x+x^2)^2 + 160(x+x^2)^3 + \dots$$

$$(2+x+x^2)^6 = 64 + 192x + 432x^2 + 640x^3 + 720x^4 + \dots$$

Coefficient of the terms in binomial expansion

To find the coefficient of a particular term in the binomial expansion of the form $(a+b)^n$, use the formula $Coef = {}^nC_r a^{n-r} b^r$ where r is the position of the term required and $r \leq n$

Example 98

Determine the coefficient of the term containing x^5 in the expansion of $(3+2x)^{15}$

Solution

The formula is $C = {}^nC_r a^{n-r} b^r$

$$C = {}^{15}C_5 (3)^{15-5} (2)^5 = 96096 \times 3^{10}$$

The coefficient is 5,674,372,704

Example 99

Determine the constant coefficient in the expansion $\left(2 - \frac{3}{4}x\right)^{11}$

Solution

The constant coefficient is the coefficient with the term containing x^0

It is easily obtained, since it is the first term of the expansion $\left(2 - \frac{3}{4}x\right)^{11}$

which is $2^{11} = 2048$

Example 100

Write down the coefficient of the constant term of the expansion

$$\left(3x - \frac{1}{4x}\right)^{10}$$

Solution

Given $\left(3x - \frac{1}{4x}\right)^{10}$. The constant term of the expansion is the term with $10 - r = 5$ thus $r = 5$

By using $C = {}^nC_r a^{n-r} b^r$

The coefficient become $C = {}^{10}C_5 (3)^5 \left(\frac{1}{4}\right)^5 = \frac{15309}{256}$

Example 101

Find the constant coefficient of the expansion $\left(3x^2 - \frac{4}{5x}\right)^6$

Solution

Given $\left(3x^2 - \frac{4}{5x}\right)^6$ then $C = {}^nC_r a^{n-r} b^r$

$$C = {}^6C_r (3x^2)^{6-r} \left(-\frac{4}{5x}\right)^r$$

$$C = {}^6C_r (3)^{6-r} (x^2)^{6-r} \left(-\frac{4}{5}\right)^r x^{-r}$$

For the constant term, x^0

$$x^{12-2r-r} = x^0 \text{ thus } r = 4$$

The coefficient is $C = {}^6C_4 (3)^2 \left(-\frac{4}{5}\right)^4 = \frac{6912}{125}$

Example 102

Find the ratio of the term having x^4 and that of x^6 in the expansion of

$$\left(4x - \frac{1}{3}\right)^{13}$$

Solution

Given $\left(4x - \frac{1}{3}\right)^{13}$

$$\begin{aligned} \text{The ratio is } \frac{C_4}{C_6} &= \frac{{}^{13}C_9(4x)^{13-9}\left(-\frac{1}{3}\right)^9}{{}^{13}C_7(4x)^{13-7}\left(-\frac{1}{3}\right)^7} = \frac{715}{1716} \times \frac{4^4 x^4}{4^6 x^6} \times \frac{\left(-\frac{1}{3}\right)^9}{\left(-\frac{1}{3}\right)^7} \\ &= \frac{715}{1716 \times 4^2 x^2 \times 3^2} = \frac{715}{247104x^2} \end{aligned}$$

The required ratio is $715:247104x^2$

Example 103

In the expansion $(3x - y)^{30}$ as a series in ascending powers of y find

- (a) The term containing x^3
- (b) The 8th term

Solution

$$(a) \quad T_{r+1} = {}^{30}C_r x^{30-r} (-y)^r$$

The term containing x^3 is the term for which $30 - r = 3$, thus $r = 27$

$$T_{28} = {}^{30}C_{27} x^3 (-y)^{27} = -4060x^3 y^{27}$$

- (b) Since the first term of the series is the term with $r = 0$, the 8th term is the term with $r = 7$.

$$T_8 = {}^{30}C_7 x^{30-7} (-y)^7 \text{ thus } T_8 = -2035800x^{23}y^7$$

Binomial theorem for any index

So far, we have discussed the binomial theorem when n is a positive integer, here we will check the expansion when n is any index.

Example 104

Use the binomial theorem to expand $\frac{1}{1+x}$ in ascending powers of x as

far as the term containing x^5

Solution

Given $\frac{1}{1+x} = (1+x)^{-1}$

From $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-1-1)x^2}{2} + \frac{(-1)(-1-1)(-1-2)x^3}{6} + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

Example 105

Obtain the expansion of $(2+x)^{-2}$ in ascending powers of x as far as the term with x^3 and determine the values of x for which the expansion is valid.

Solution

Refer the general expansion $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \dots$

Using the expansion above, we get

$$(2+x)^{-2} = 2^{-2} + (-2)(2)^{-3}x + \frac{(-2)(-3)(2)^{-4}x^2}{2!} + \frac{(-2)(-3)(-4)(2)^{-5}x^3}{3!} + \dots$$

$$(2+x)^{-2} = \frac{1}{4} - \frac{1}{4}x + \frac{3}{16}x^2 - \frac{1}{8}x^3 + \dots$$

To determine the validity of the expansion, compare the expansion with

$$(a+b)^n = a^n \left(1 + \frac{b}{a}\right)^n$$

$$(2+x)^{-2} = \left(2\left(1 + \frac{1}{2}x\right)\right)^{-2} = 2^{-2} \left(1 + \frac{1}{2}x\right)^{-2}$$

From the expression above, for the expansion to be valid, use the interval $-1 < \frac{b}{a} < 1$, for this case $-1 < \frac{1}{2}x < 1$ which leads to $-2 < x < 2$.

Example 106

Find the first 4 terms of the expansion $\sqrt[3]{2-3x}$ using binomial theorem and state the values of x for which the expansion is valid.

Solution

$$\text{From } (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \dots$$

$$\sqrt[3]{2-3x} = (2-3x)^{\frac{1}{3}} = 2^{\frac{1}{3}} + \frac{1}{3} \left(\frac{1}{2^{\frac{2}{3}}} \right) (-3x) + \frac{\frac{1}{3} \left(\frac{1}{3} - 1 \right) \left(2^{\frac{1}{3}-2} \right) (-3x)^2}{2!}$$

$$\sqrt[3]{2-3x} = 2^{\frac{1}{3}} - 2^{\frac{2}{3}}x - 2^{\frac{5}{3}}x^2 - \frac{5}{3} \times 2^{\frac{8}{3}}x^3 - \dots$$

Consider, $(2-3x)^{\frac{1}{3}} = 2^{\frac{1}{3}} \left(1 - \frac{3x}{2} \right)^{\frac{1}{3}}$, the validity of the expansion holds

when $-1 < \frac{3x}{2} < 1$ therefore the expansion is valid when $-\frac{2}{3} < x < \frac{2}{3}$,

thus $|x| < \frac{2}{3}$

6.13 Partial fractions

A rational function is a fraction in which both the numerator and denominator are polynomials. The function, which can be expressed as

$f(x) = \frac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ are both polynomials. Partial fractions is

the decomposition of fractions into sum of other simplified fractions.

Proper fraction with linear factors in denominator

Consider the function $\frac{p(x)}{(x-a)(x-b)\dots(x-z)}$ where $a, b, c, \dots, z$ are constants, then the function can be expressed as the sum of the simplified rational functions as,

$\frac{p(x)}{(x-a)(x-b)\dots(x-z)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \dots + \frac{Z}{x-z}$ where $A, B, C, \dots, Z$ are real numbers.

$$p(x) = (x-a)(x-b)(x-c)\dots(x-z) \left(\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \dots + \frac{Z}{x-z} \right).$$

Example 107

Express $\frac{2}{(x-1)(x+2)}$ into partial fractions

Solution

$$\begin{aligned} \text{Given } \frac{2}{(x-1)(x+2)} &= \frac{A}{x-1} + \frac{B}{x+2} \\ 2 &= A(x+2) + B(x-1) \end{aligned}$$

When $x = 1$

$$2 = 3A \Rightarrow A = \frac{2}{3}$$

When $x = -2$

$$2 = -3B \Rightarrow -\frac{2}{3}$$

$$\text{Therefore, } \frac{2}{(x-1)(x+2)} = \frac{2}{3(x-1)} - \frac{2}{3(x+2)}$$

Example 108

Resolve $\frac{x-3}{x^2+x-6}$ as partial fractions

Solution

Factorize the denominator $x^2 + x - 6 = (x-2)(x+3)$

$$\frac{x-3}{x^2+x-6} = \frac{x-3}{(x-2)(x+3)}$$

$$\frac{x-3}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

$$x-3 = A(x+3) + B(x-2)$$

Compare the exponents $\begin{cases} A+B=1 \\ 3A-2B=-3 \end{cases}$ solve simultaneously

Then, $A = -\frac{1}{5}$ and $B = \frac{6}{5}$

Therefore, $\frac{x-3}{x^2+x-6} = \frac{6}{5(x+3)} - \frac{1}{5(x-2)}$

Example 109

Express $\frac{x^2-3x+1}{2x^3+5x^2-6x-9}$ as partial fractions

Solution

Factorizing $2x^3+5x^2-6x-9 = (x+1)(2x-3)(x+3)$

$$\frac{x^2-3x+1}{2x^3+5x^2-6x-9} = \frac{x^2-3x+1}{(x+1)(2x-3)(x+3)}$$

$$\frac{x^2-3x+1}{(x+1)(2x-3)(x+3)} = \frac{A}{x+1} + \frac{B}{2x-3} + \frac{C}{x+3}$$

$$x^2-3x+1 = A(2x-3)(x+3) + B(x+1)(x+3) + C(x+1)(2x-3)$$

When $x = -1$

$$5 = A(-5)(2) \Rightarrow A = -\frac{1}{2}$$

When $x = -3$

$$19 = C(-2)(-9) \quad \text{then} \quad C = \frac{19}{18}$$

When $x = 1$

$$1 - 3 + 1 = A(-1)(4) + B(2)(4) + C(2)(-1)$$

$$-1 = -4A + 8B - 2C$$

But $A = -\frac{1}{2}$ and $C = \frac{19}{18}$ thus $-4\left(-\frac{1}{2}\right) + 8B - 2\left(\frac{19}{18}\right) = -1$

$$2 + 8B - \frac{19}{9} = -1 \quad \text{then} \quad B = -\frac{1}{9}$$

Therefore, $\frac{x^2 - 3x + 1}{(x+1)(2x-3)(x+3)} = -\frac{1}{2(x+1)} - \frac{1}{9(2x-3)} + \frac{19}{18(x+3)}$

Example 110

Express $\frac{3}{x-2} + \frac{2}{x+1}$ as a single fraction

Solution

This can be done as follows,

$$\frac{3}{x-2} + \frac{2}{x+1} = \frac{3(x+1) + 2(x-2)}{(x-2)(x+1)} = \frac{5x-1}{x^2-x-2}$$

Proper fraction with quadratic factors in denominator

Consider, $\frac{p(x)}{(ax^2+bx+c)(x-k)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x-k}$. This is normally applied when a quadratic root cannot be factorized into real numbers or has no simple roots. If the quadratic factor can be factorized, factorize and use the above technique to partialize.

Example 111

Partialize $\frac{x^2 + 4}{(x^2 - 3x + 1)(x - 2)}$

Solution

$$\frac{x^2 + 4}{(x^2 - 3x + 1)(x - 2)} = \frac{Ax + B}{x^2 - 3x + 1} + \frac{C}{x - 2}$$

$$x^2 + 4 = (Ax + B)(x - 2) + C(x^2 - 3x + 1)$$

$$x^2 + 4 = Ax^2 - 2Ax + Bx - 2B + Cx^2 - 3Cx + C$$

Compare:

$$\begin{cases} Ax^2 + Cx^2 = x^2 \\ -2Ax + Bx - 3Cx = 0 \\ -2B + C = 4 \end{cases}$$

The coefficients; $\begin{cases} A + C = 1 \\ -2A + B - 3C = 0 \\ -2B + C = 4 \end{cases}$

Solve for A , B and C : $A = 9$, $B = -6$ and $C = -8$

Therefore, $\frac{x^2 + 4}{(x^2 - 3x + 1)(x - 2)} = \frac{9x - 6}{x^2 - 3x + 1} - \frac{8}{x - 2}$

Example 112

Express $\frac{x - 5}{x^3 - 2x^2 + x - 2}$ as a partial fraction

Solution

Factorize, $x^3 - 2x^2 + x - 2 = (x^2 + 1)(x - 2)$

$$\frac{x-5}{(x^2+1)(x-2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

$$x-5 = A(x^2+1) + (Bx+C)(x-2)$$

$$x-5 = Ax^2 + A + Bx^2 - 2Bx + Cx - 2C$$

$$\text{Compare the coefficients } \begin{cases} A+B=0 \\ -2B+C=1 \\ A-2C=-5 \end{cases}$$

$$\text{Then } A = -\frac{3}{5}, B = \frac{3}{5} \text{ and } C = \frac{11}{5}$$

$$\text{Therefore, } \frac{x-5}{(x^2+1)(x-2)} = -\frac{3}{5(x-2)} + \frac{3x+11}{5(x^2+1)}$$

Example 113

$$\text{Partialize } \frac{x^2+2x+3}{x^4+x^2-2}$$

Solution

$$\text{Factorize, } x^4+x^2-2 = (x^2-1)(x^2+2) = (x-1)(x+1)(x^2+2)$$

$$\frac{x^2+2x+3}{(x-1)(x+1)(x^2+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+2}$$

$$x^2+2x+3 = A(x+1)(x^2+2) + B(x-1)(x^2+2) + (Cx+D)(x^2-1)$$

$$\text{When } x=1 \Rightarrow A=1$$

$$\text{When } x=-1 \Rightarrow B = -\frac{1}{3}$$

$$\text{When } x=0$$

$$3 = 2A - 2B - D \text{ thus } D = -\frac{1}{3}$$

$$\text{When } x=2$$

$$11 = 18A + 6B + 6C + 3D \text{ thus } C = -\frac{2}{3}$$

Therefore, $\frac{x^2 + 2x + 3}{x^4 + x^2 - 2} = \frac{1}{x-1} - \frac{1}{3(x+1)} - \frac{2x+1}{3(x^2+2)}$

Proper fraction with repeated factors in denominator

Consider a rational function,

$$\frac{p(x)}{(x+a)^n(x+b)} = \frac{A}{x+a} + \frac{B}{(x+a)^{n-1}} + \frac{C}{(x+a)^{n-2}} + \dots + \frac{W}{(x+a)^n} + \frac{Z}{x+b}$$

Where $A, B, C, \dots, W$ and Z are real constants.

Example 114

Partialize $\frac{x-3}{(x+1)^2(x-2)}$

Solution

Given $\frac{x-3}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$

$$x-3 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$$

$$x-3 = A(x^2 - x - 2) + Bx - 2B + C(x^2 + 2x + 1)$$

$$x-3 = Ax^2 - Ax - 2A + Bx - 2B + Cx^2 + 2Cx + C$$

$$\begin{cases} A+C=0 \\ -A+B+2C=1 \\ -2A-2B+C=-3 \end{cases}$$

$$A = \frac{1}{9}, B = \frac{4}{3} \text{ and } C = -\frac{1}{9}$$

Therefore, $\frac{x-3}{(x+1)^2(x-2)} = \frac{1}{9(x+1)} + \frac{4}{3(x+1)^2} - \frac{1}{9(x-2)}$

Example 115

Express $\frac{x^2 + 3x + 5}{4x^3 - 8x^2 + 5x - 1}$ to partial fractions

Solution

$$\frac{x^2 + 3x + 5}{4x^3 - 8x^2 + 5x - 1} = \frac{x^2 + 3x + 5}{(x-1)(2x-1)^2}$$

$$\frac{x^2 + 3x + 5}{(x-1)(2x-1)^2} = \frac{A}{x-1} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$$

$$x^2 + 3x + 5 = A(2x-1)^2 + B(x-1)(2x-1) + C(x-1)$$

$$x^2 + 2x + 5 = 4Ax^2 - 4Ax + A + 2Bx^2 - 3Bx + B + Cx - C$$

$$\text{Coefficients, } \begin{cases} 4A + 2B = 1 \\ -4A - 3B + C = 2 \\ A + B - C = 5 \end{cases}$$

$$A = 8, B = -\frac{31}{2} \text{ and } C = -\frac{25}{2}$$

$$\text{Therefore, } \frac{x^2 + 3x + 5}{(x-1)(2x-1)^2} = \frac{8}{x-1} - \frac{31}{2(2x-1)} - \frac{25}{2(2x-1)^2}$$

Example 116

Partialize $\frac{x^2 - 3}{(x-4)(x+1)^3}$

Solution

$$\frac{x^2 - 3}{(x-4)(x+1)^3} = \frac{A}{x-4} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

$$x^2 - 3 = A(x+1)^3 + B(x-4)(x+1)^2 + C(x-4)(x+1) + D(x-4)$$

$$\text{When } x = 4 \text{ then } A = \frac{13}{125}$$

When $x = -1$ then $D = \frac{2}{5}$

When $x = 0$
 $A - 4B - 4C - 4D = -3$

When $x = 1$
 $8A - 12B - 6C - 3D = -2$

Solve
$$\begin{cases} 4B + 4C = \frac{188}{125} \\ 12B + 6C = \frac{204}{125} \end{cases}$$

$$B = -\frac{13}{125}, C = \frac{12}{25}$$

$$\frac{x^2 - 3}{(x-4)(x+1)^3} = \frac{13}{25(x-4)} - \frac{13}{125(x+1)} + \frac{12}{25(x+1)^2} + \frac{2}{5(x+1)^3}$$

Improper fractions

When a rational function is improper fraction, before partializing, divide first and partialize the remainder.

Consider $f(x) = \frac{p(x)}{g(x)}$ where, $p(x) > g(x)$, $\frac{p(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$, here we

partialize $\frac{r(x)}{g(x)}$

Example 117

Partialize $\frac{x^3 + 3x^2 + 4x - 2}{x^2 + x - 42}$

Solution

Divide by long division method, $\frac{x^3 + 3x^2 + 4x - 2}{x^2 + x - 42}$

$$\begin{array}{r}
 x^2 + x - 42 \overline{) x^3 + 3x^2 + 4x - 2} \\
 \underline{x^3 + x^2 - 42x} \\
 2x^2 + 46x - 2 \\
 \underline{2x^2 + 2x - 84} \\
 44x + 82
 \end{array}$$

Therefore, $\frac{x^3 + 3x^2 + 4x - 2}{x^2 + x - 42} = x + 2 + \frac{44x + 82}{x^2 + x - 42}$

Partialize $\frac{44x + 82}{x^2 + x - 42}$

$$\frac{44x + 82}{(x - 6)(x + 7)} = \frac{A}{x - 6} + \frac{B}{x + 7}$$

$$44x + 82 = A(x + 7) + B(x - 6)$$

$$44x + 82 = Ax + 7A + Bx - 6B$$

$$\begin{cases} A + B = 44 \\ 7A - 6B = 82 \end{cases} \text{ thus } A = \frac{346}{13} \text{ and } B = \frac{226}{13}$$

$$\frac{346}{13(x - 6)} + \frac{226}{13(x + 7)}$$

Therefore, $x + 2 + \frac{44x + 82}{(x - 6)(x + 7)} = x + 2 + \frac{346}{13(x - 6)} + \frac{226}{13(x + 7)}$

Partial fraction to find the sum of the series

Some series need to be expressed into partial fractions to get their sum. These series are the one expressed in rational form. Consider the following examples

Example 118

Express $\frac{1}{n(n-1)}$ as partial fraction and show that

$$\frac{1}{n(n-1)} = \frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \frac{1}{4 \times 3} + \dots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$$

Solution

$$\text{Let } \frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$$

When $n=1$ then $B=1$ and when $n=2$ then $A=-1$

$$+ \left\{ \begin{array}{l} A_2 = \frac{1}{1} - \frac{1}{2} \\ A_3 = \frac{1}{2} - \frac{1}{3} \\ A_4 = \frac{1}{3} - \frac{1}{4} \\ A_5 = \frac{1}{4} - \frac{1}{5} \\ \dots\dots\dots \\ A_{n-2} = \frac{1}{n-3} - \frac{1}{n-2} \\ A_{n-1} = \frac{1}{n-2} - \frac{1}{n-1} \\ A_n = \frac{1}{n-1} - \frac{1}{n} \end{array} \right.$$

$$\text{Hence } S_n = 1 - \frac{1}{n}$$

Example 119

Find the sum of $\frac{4n}{(4n^2 - 1)(2n + 3)}$

Solution

Partial fraction, $\frac{4n}{(4n^2 - 1)(2n + 3)} = \frac{4n}{(2n - 1)(2n + 1)(2n + 3)}$

$$= \frac{4n}{(2n - 1)(2n + 1)(2n + 3)} = \frac{A}{2n - 1} + \frac{B}{2n + 1} + \frac{C}{2n + 3}$$

$$4n = A(2n + 1)(2n + 3) + B(2n - 1)(2n + 3) + C(2n - 1)(2n + 1)$$

Since the equation is true for all n , when $n = 0$ we get $3A - 3B - C = 0$

When $n = 1$ we get

$$4n = A(2n + 1)(2n + 3) + B(2n - 1)(2n + 3) + C(2n - 1)(2n + 1)$$

Then $15A + 5B + 3C = 4$

When $n = -1$

$$4n = A(2n + 1)(2n + 3) + B(2n - 1)(2n + 3) + C(2n - 1)(2n + 1)$$

Then $-A - 3B + 3C = -4$

$$\begin{cases} 3A - 3B - C = 0 \\ 15A + 5B + 3C = 4 \\ -A - 3B + 3C = -4 \end{cases}$$

Solve for A , B and C , then $A = \frac{1}{4}$, $B = \frac{1}{2}$ and $C = -\frac{3}{4}$

$$\frac{4n}{(2n - 1)(2n + 1)(2n + 3)} = \frac{1}{4(2n - 1)} + \frac{1}{2(2n + 1)} - \frac{3}{4(2n + 3)}$$

$$\text{Let } A_n = \frac{1}{4(2n - 1)} + \frac{1}{2(2n + 1)} - \frac{3}{4(2n + 3)}$$

$$\begin{aligned}
 & \left\{ \begin{aligned} A_1 &= \frac{1}{4} + \frac{1}{6} - \frac{3}{20} \\ A_2 &= \frac{1}{12} + \frac{1}{10} - \frac{3}{28} \\ A_3 &= \frac{1}{20} + \frac{1}{14} - \frac{3}{26} \\ & \dots\dots\dots \\ A_{n-2} &= \frac{1}{8n-20} + \frac{1}{4n-6} - \frac{3}{8n-4} \\ A_{n-1} &= \frac{1}{8n-12} + \frac{1}{4n-2} - \frac{3}{8n+4} \\ A_n &= \frac{1}{8n-4} + \frac{1}{4n+2} - \frac{3}{8n+12} \end{aligned} \right. \\
 & + \dots\dots\dots \\
 & S_n = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} + \frac{1}{4n+2} - \frac{3}{8n+4} - \frac{3}{8n+12} \\
 & S_n = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} + \frac{1}{4n+2} - \frac{3}{8n+4} - \frac{3}{8n+12}
 \end{aligned}$$

Therefore $S_n = \frac{1}{2} - \frac{10n+9}{(4n+2)(8n+12)}$

Example 120

Express $\frac{1}{n(n+2)}$ into partial fractions, hence deduce that

$$\frac{1}{1 \times 3} + \frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \dots + \frac{1}{n(n+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$$

Solution

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$An + 2A + Bn = 1$$

Therefore $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ thus $\frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}$

$$A_n = \frac{1}{2n} - \frac{1}{2(n+2)}$$

$$+ \left\{ \begin{array}{l} A_1 = \frac{1}{2} - \frac{1}{6} \\ A_2 = \frac{1}{4} - \frac{1}{8} \\ A_3 = \frac{1}{6} - \frac{1}{10} \\ \dots\dots\dots \\ A_{n-2} = \frac{1}{2n-4} - \frac{1}{2n} \\ A_{n-1} = \frac{1}{2n-2} - \frac{1}{2n+2} \\ A_n = \frac{1}{2n} - \frac{1}{2n+4} \end{array} \right.$$

$$S_n = \frac{1}{2} + \frac{1}{4} - \frac{1}{2n+2} - \frac{1}{2n+4}$$

Therefore,
$$S_n = \frac{3}{4} - \frac{2n+3}{2(n+2)(n+1)}$$

MISCELENEOUS EXERCISE: ALGEBRA

Partialize the following

$$1. \frac{x^2 + x + 2}{(x+4)(x^2+2)^2}$$

$$2. \frac{x+4}{(x^2+3)(x+1)}$$

$$3. \frac{x-3}{x^2+4x+3}$$

$$4. \frac{x^2+3x}{x^3+6x^2+11x+6}$$

$$5. \frac{3-x}{(x^2+3)(x+3)}$$

$$6. \frac{x-11}{x^2-x-2}$$

$$7. \frac{x^3-6x+9}{x^2+x-2}$$

$$8. \frac{x+10}{(x-1)(x+2)^3}$$

$$9. \frac{x^2+5}{x^3+x^2+x+1}$$

$$10. \frac{x^2+x+1}{6x^3-5x^2-2x+1}$$

$$11. \frac{3x+7}{(6x^2+7x+1)(x^2+1)}$$

$$12. \frac{3x+1}{6x^2-11x-7}$$

$$13. \frac{x^3+6x^2+6}{(2x^2+11x+5)(3x^2-10x+3)}$$

$$14. \frac{4-5x}{12x^3+6x^2-12x-6}$$

$$15. \text{ f } \frac{ax+b}{3+x-2x^2} = \frac{4}{3-2x} + \frac{5}{x+1} \text{ find the value of } a, \text{ and } b.$$

$$16. \frac{x^4-5x^3+6x^2-7}{(x-1)(x-3)}$$

$$17. \frac{2x+9}{x^2-x-6}$$

$$18. \frac{x^2+2}{x(x+1)(x+2)}$$

$$19. \frac{9x^3-x}{3x^2+8x-3}$$

$$20. \frac{x^3-x^2-2}{x^2+x}$$

$$21. \frac{x^2-x+4}{x^3-3x^2+2x}$$

$$22. \frac{1+x^2}{(1+x)(1+x^3)}$$

$$23. \frac{5x^3+2x^2+5x}{x^4-1}$$

$$24. \frac{x^2-x-1}{x^3-8}$$

$$25. \frac{2x^3-x^2-7}{x^4+3x^2}$$

$$26. \frac{4x}{x^4-16}$$

$$27. \frac{3x^2}{(2x-1)^3}$$

$$28. \frac{10-11x}{(x-4)(x^2+1)}$$

$$29. \frac{2x^2-11x+5}{(x^2+2x-5)(x-3)}$$

30. Express $\frac{1}{(n+1)(n+3)}$ in partial fractions and hence find the sum of

$$\frac{1}{2 \times 4} + \frac{1}{3 \times 5} + \frac{1}{4 \times 6} + \dots + \frac{1}{(n+1)(n+3)}.$$

31. Express $\frac{1}{x(x+1)}$ in partial fractions and show that $\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} \right) = 1$

32. Find the sum of the series $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$

33. Show that $\sum_{n=1}^n \frac{3}{(3n-1)(3n+2)} = \frac{1}{2} - \frac{1}{3n+2}$

34. Find the sum of the series

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \frac{1}{11 \times 14} + \dots + \frac{1}{9n^2 + 3n - 2}$$

35. Use partial fraction to show that

$$\frac{2}{3 \times 5} + \frac{3}{3 \times 5 \times 7} + \frac{4}{5 \times 7 \times 9} + \dots + \frac{n+1}{(4n^2-1)(3n+3)} = \frac{n(5n+7)}{6(2n+1)(2n+3)},$$

state whether the series $\sum_{n=1}^n \frac{n+1}{(4n^2-1)(3n+3)}$ converge as $n \rightarrow \infty$

and it does find $\sum_{n=1}^{\infty} \frac{n+1}{(4n^2-1)(3n+3)}$ (sum to infinity)

36. Show that $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

37. Write down the expansion of $(1-x)^{-2}$ in ascending powers of x and

deduce that Show that $\sum_{n=0}^{\infty} (a+bn)x^n = \frac{a+(b-a)x}{(1-x)^2}$ for $|x| < 1$

hence prove that $\sum_{n=0}^{\infty} \frac{8n+1}{3^{2n}} = \frac{9}{4}$.

38. Find the sum of the series

$$\frac{5}{1 \times 2 \times 3} + \frac{8}{2 \times 3 \times 4} + \frac{11}{3 \times 4 \times 5} + \dots + \frac{3n+2}{n(n+1)(n+2)}$$

39. Express into partial fractions $\frac{4+x}{(1+x)(2-x)}$

40. Partialize $\frac{x-4}{x+3}$

41. Partialize $\frac{x(3x+5)}{(2x+1)(x^2-4)}$

42. Partialize $\frac{6-x^2}{4-x^2}$

43. Partialize $\frac{2x+3}{(x-1)(x^2+4)}$

44. Express $\frac{4x+5}{(2x+1)^2}$ as a partial fractions

45. Express $\frac{x+1}{(x-1)(x+2)^3}$ as a sum of partial fractions

46. Express $\frac{5x^2-3}{x^2(x+1)}$ as the sum of partial fractions

47. Express $\frac{2x+7}{(x-1)(x+2)}$ in partial fractions and hence find the first three terms of its binomial expansion

48. Find the first three terms in ascending powers of x in the binomial expansion of $\frac{4}{(1-3x)(1-x)^2}$

49. Find the expansion of $\frac{1}{1-2x}$ in the form $a+bx+cx^2+dx^3+\dots$ and hence determine the values of a, b, c and d .

50. Expand $(2-x)(1+x)$ and hence express $\frac{3x}{2+x-x^2}$ in partial fractions
51. Show that the sum of $1+5x+9x^2+13x^3+\dots$ is $\frac{1+3x}{(1-x)^2}$ if $|x|<1$
52. Find the sum of the series $3+5n+7n^2+\dots$ if $|n|<1$
53. Show that $2^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times \dots = 2$
54. (a) Solve for x and y the following system of equations, give your answers correct to three significant figures $\begin{cases} 2^x 3^y = 5 \\ 4^x = 7(9^y) \end{cases}$
- (b) Let $a = -(b+c)$ or otherwise, prove that $(a+b+c)$ is a factor of $a^3+b^3+c^3-3abc$. Deduce a factor of $8x^3+y^3-z^3+6xyz$
55. Show that if $a > 0$ the function $f(x) = ax^2 + bx + c$ has a least value, and find it. Hence find the values of a , b and c , if the function $f(x) = ax^2 + bx + c$ has the minimum value -5 when $x = -1$ and the value zero when $x = -2$.
56. (a) Find the value of a and n if the first three terms of the binomial expansion of $(1+ax)^n$ are $1-4x+\frac{20}{3}x^2+\dots$
- (b) Find the expansion in ascending powers of x , as far as the term in x^2 , of the expression $\frac{3+x}{4-x^2+\sqrt[3]{8-2x^2}}$
- (c) Prove that unless $x=1$ the sum of the series $S: 1+x+x^2+\dots+x^{n-1}$ is $\frac{1-x^n}{1-x}$
- (d) By multiplying the series $S': 1+2x+3x^2+\dots+nx^{n-1}$ by $(1-x)$ and summing the resulting series, find the sum of the series S'
- (e) Consider what happens to the sum of each of the given series as $n \rightarrow \infty$, if $0 < x < 1$. (It may be assumed that if $0 < x < 1$ then $nx^n \rightarrow 0$ as $n \rightarrow \infty$)
57. (a) Find the coefficient of $\frac{1}{x}$ in the expansion of $\left(2x - \frac{1}{x}\right)^5$

- (b) Find the coefficient of x^3 in the expansion of
- (i) $(1+2x)^6$ (ii) $(1-3x)(1+2x)^6$
- (c) If the polynomial $x^4 - 2x^3 - 2x^2 + a$ is divisible by $x^2 - 4x + 4$.
- (i) Find the value of a .
- (ii) When a has this value, show that $x^4 - 2x^3 - 2x^2 + a$ is never negative.
58. Let $f(x) = \frac{9x^2 + 4}{(2x+1)(x-2)^2}$
- (a) Express $f(x)$ in partial fractions
- (b) Show that, when x is sufficiently small for x^3 and higher powers to be neglected, $f(x) = 1 - x + 5x^2$
59. A geometric progression has first term 64 and sum to infinity 256. Find
- (a) The common ratio,
- (b) The sum of the first ten terms.
60. (a) If $2^x = 5^y$ show that $\log_2 5 = \frac{x}{y}$
- (b) Solve the equation $2x^3 + ax^2 - 13x - 6$ if it is divisible by $x - 3$
- (c) Solve the inequality $|2x+1| < |x|$
61. The polynomial $x^3 - x^2 + ax + b$ is divisible by $x+1$ and leaves a remainder of 12 when divided by $x-2$. Find
- (a) The values of a and b
- (b) When a and b have these values, factorize the polynomial.
62. (a) Expand $(1+4x)^{-\frac{1}{2}}$ in ascending powers of x , up to and including the term in x^3 , simplify the coefficients.
- (b) The first three terms in the expansion of $(2+ax)^n$, in ascending powers of x , are $32 - 40x + bx^2$. Find the values of the constant n , a , and b .
63. (a) Express $f(x) = \frac{x^2 + 7x - 6}{(x-1)(x-2)(x+1)}$ in partial fractions
- (b) Show that, when x is sufficiently small for x^4 and higher powers to be neglected $f(x) = -3 + 2x - \frac{3}{2}x^2 + \frac{11}{4}x^3$.

64. (a) Find the first 3 terms in the expansion of $(2-x)^6$ in ascending powers of x .
- (c) Find the value of k for which there is no term in x^2 in the expansion of $(1+kx)(2-x)^6$
65. The series of natural numbers is grouped as follows: (1), (2,3,4), (5,6,7,8,9), ...
- (a) Find the total number of integers in the first $n-1$ brackets
- (b) Show that the first number in the n^{th} brackets is $n^2 - 2n + 2$
- (c) Show that the sum of the numbers in the n^{th} bracket is $n^3 + (n-1)^3$
- (d) If the first number in the n^{th} bracket is denoted by a and the first number in the $(n+1)^{\text{th}}$ bracket is denoted by b , show that the sum of the numbers in the n^{th} brackets is exactly divisible by $-(a-b)$ and that the quotient is an odd number.
66. (a) Find the least positive integral value of n for which $\left(\frac{4}{5}\right)^n = \left(\frac{1}{25}\right)$
- (b) Find the value of k for which the equation $(a-x)(x-b) = k$ has repeated roots.
67. (a) Expand $(1+x)^p$ and $\frac{1+ax}{1+bx}$ in ascending powers of x as far as the terms in x^3
- (b) If these expansions are identical as far as the terms in x^2 express a and b in terms of p .
- (c) By taking $p = \frac{2}{3}$ and $x = \frac{1}{8}$ show that an approximation to the cube root of 81 is $\frac{212}{49}$
- (d) Find in terms of p and x , the difference between the terms in x^3 of the two expansions, and evaluate this difference, correct to one significant figure, when $p = \frac{2}{3}$ and $x = \frac{1}{8}$

68. The equation $f(x) = 2x^4 + ax^3 + bx^2 - 4x - 4$ where a and b are real constants. Given that $f(-\frac{1}{2}) = 0$ and $f(2) = 0$, find the values of a and b .
- Express $f(x)$ as the product of factors and hence solve for x .
 - Find the set of values of x for which $f(x) > 0$
69. (a) Express $\frac{3x}{(x-1)(x+2)}$ in partial fractions
- (b) Evaluate $\int \frac{3x}{(x-1)(x+2)} dx$
70. Find constants a and b so that in the expansion of the function $x + a \sin x + b x \cos x$ in ascending powers of x , the first non-zero terms is the one in x^5 , and determine the coefficient of that term.
71. (a) Find the set $|x+3| > 2|x-3|$
- (b) The ordered positive integers are bracketed thus: (1); (2,3); (4,5,6); ... there being n th bracket. Find simplified expressions for
- The last integer in the n th bracket,
 - The first integer in the n th bracket.
72. Prove that each of the circles $x^2 + y^2 - 4x = 0$ and $x^2 + y^2 - 12x - 8y + 43 = 0$ lies completely outside the other.
73. A sequence with n th term u_n is defined by $u_1 = 3$ and $u_{n+1} = u_n + 4$ for $n \geq 1$
- Write down the values of u_2 , u_3 and u_4
 - By identifying the type of the sequence or otherwise, express u_n in terms of n
74. (a) Solve the equations $x + 2y = 9$ and $xy + 18 = 0$ simultaneously and find the distance between the points of intersections.
- (b) The progression has the first term of 12 and a fifth term of 18.
- Find the sum of the first 31 terms if the progression is arithmetic progression
 - Find the difference between the sum of the first 10 terms and the last 10 terms if the progression is geometric with 40 terms.

75. Solve $|y-1| < 6$ hence solve the inequality $|2^x - 1| < 6$ expressing any numerical value in the form of $\frac{\ln p}{\ln q}$ where p and q are integers.
76. (a) Solve the inequality $|x+2| < |5-2x|$
 (b) One root of the cubic equation $3x^3 + ax^2 - 2x - 8$ is -2 , find the value of a and factorize the equation completely, hence solve the equation.
77. The polynomial $x^4 + 4x^3 + x + a$ has a factor $x^2 + x + 2$ find the value of a and the factorize the expression completely.
78. (a) Expand $(1-3x)^{-\frac{1}{3}}$ in ascending powers of x , up to the term in x^3 , simplifying the coefficients
 (b) Use the results above to find the value of $\sqrt[3]{0.4}$ correct to 3 significant figures.
79. Express $4x^3 + 60x^2 + 299x + 495$ in the form of $A(x+B)^3 + C(x+B)$, find the integers A , B and C , hence solve the equation $9(y-2)^3 - 4(y-2) = 0$
80. (a) Solve the inequality $|9-2x| < 1$
 (b) Solve the equation $\log(x+5) = 2 - \log x$
 (c) Express $f(x) = \frac{6+7x}{(2-x)(1+x^2)}$ in partial fractions
 (d) Show that when x is sufficient small for x^4 and higher powers to be neglected, $f(x) = 3 + 5x - \frac{1}{2}x^2 - \frac{15}{4}x^3$
81. Find the value of the term which is independent of x in the expansion of $\left(x + \frac{3}{x}\right)^4$
82. A geometric progression, for which the common ratio is positive, has a second term of 18 and a fourth term of 8. Find
 (a) The first terms and the common ratio of the progression
 (b) The sum to infinity of the progression
83. (a) Express $2x^2 + 8x - 10$ in the form $a(x+b)^2 + c$
 (b) For the curve $y = 2x^2 + 8x - 10$, state the least value of y and the corresponding value of x .

- (c) Find the set of values of x for which $y \geq 14$
84. (a) Solve the inequality $|2x - 1| < |3x|$
- (b) The cubic polynomial $f(x) = 2x^3 + ax^2 + b$ it is given that $(x + 1)$ is a factor of $f(x)$, and that when $f(x)$ is divided by $(x + 2)$ the remainder is -5 . Find the values of a and b .
- (c) Express 9^x in terms of y , where $y = 3^x$
- (d) Hence solve the equation $2(9^x) - 7(3^x) + 3 = 0$ expressing your answers for x in terms of logarithm where appropriate.
85. Solve the following equations
- (a) $e^x(2e^x - 1) = 10$
- (b) $\log_5(8y - 6) - \log_5(y - 5) = \log_4 16$
86. The cubic polynomial $f(x)$ is such that the coefficient of x^3 is -1 and the roots of the equation $f(x) = 0$ are $1, 2$ and k . Given that $f(x)$ has a remainder of 8 when divided by $x - 3$, find
- (a) The value of k
- (b) The remainder when $f(x)$ is divided by $x + 3$
87. (a) Find, in its simplest form, the product of $a^{\frac{1}{3}} + b^{\frac{2}{3}}$ and $a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$
- (b) Give that $(2^{2x+2})(5^{x-1}) = k$ evaluate 10^x
88. (a) Simplify $\frac{16^{x+1} + 20(4^{2x})}{2^{x-2}8^{x+2}}$
- (b) Find the solution set of the quadratic inequality
- (i) $x^2 - 8x + 12 > 0$
- (ii) $x^2 - 8x < 0$
- (iii) Find the solution set of the inequality $|x^2 - 8x + 6| < 6$
89. The equation $8x^3 + 12x^2 + 4x - 1 = 0$ has roots α, β, γ . Show that the equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$ is $y^3 - y - 1 = 0$
90. Prove by induction that, for all $N \geq 1$, $\sum_{n=1}^N \frac{n+2}{n(n+1)2^n} = 1 - \frac{1}{(n+1)2^n}$
91. Use the relevant standard results to find, $S_n = \sum_{n=1}^n (8n^3 - 6n^2)$

92. (a) Show that if $\log a + \log c = 2\log b$ then a , b , and c are in geometric progression.

(b) Show that if $\log x + \log y = 3\log y$ then x , y^3 and yz are in geometric progression

(c) Show that $x - y - z$ is a factor of the expression $x^3 + y^3 + z^3 - yz(y + z) - zx(z + x) - xy(x + y) + 2xyz$. Without further working write down two other factors of this expression.

93. The first three terms in the expansion of $\left(a + \frac{x}{4}\right)^5$ are $32 + bx + cx^2$.

Find the value of each out the constants a , b , c .

94. (a) Given that $X = \begin{pmatrix} a & -1 \\ 2 & -3 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 & 1 \\ 4 & 3a \end{pmatrix}$ if a is a constant, find the value of a if $\det(X) = 4\det(Y) + 1$
- (b) If $\log_b a^2 = 3$ find $\log_a b^2$

95. (a) Solve the simultaneous equations
$$\begin{cases} x + 2y - z = 2 \\ 2x - y + z = 0 \\ \lambda x + 5y - 3(1 + \lambda)z = 4 \end{cases}$$

(b) If p , q and r form an arithmetic progression. Show that their squares cannot form an arithmetic progression, unless their common difference is zero.

- (i) When $\lambda \neq 0$
(ii) When $\lambda = 0$

96. (a) Use standard results to show that $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$ and

deduce the sum of the first n terms of the series $1^2 + 3^2 + 5^2 + \dots$

(b) Inside a given sector of radius 8 cm and central angle 30° a square is drawn such that each vertex is either on one radius or on the arc. Determine the area of the sector which lies outside the square.

97. (a) Sum to n terms, and to infinity, the series
$$\frac{2}{1 \times 3 \times 5} + \frac{4}{3 \times 5 \times 7} + \frac{6}{5 \times 7 \times 9} + \dots$$

- (b) Given two concentric circles C_1 and C_2 of radius 9 cm and 7 cm respectively. A straight-line cuts C_1 at P and S and C_2 at Q and R , such that P, Q, R and S are consecutive points of the line. If $PQ=QR=RS$, determine
- The length of the chord
 - The length of the small arc PS
98. (a) Solve the following equation $\log_2 x - \log_8 x = 4$
- (b) Find the roots of the quadratic equation $x^2 - 6x + k = 0$ if one the root is 3 times the other and hence find the value of the constant k .
- (c) Expand $\frac{5}{(1+3x)(-2x+1)}$ as the series of ascending power of x up to the fourth term and hence find the value of x for which the expression is not valid
- (d) Prove by mathematical induction that $3^{2n} - 1$ is divisible by 4 for all positive integer values.
99. (a) By induction, show that $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2(2n^2 - 1)$
- (b) Expand $\frac{2 + \left(1 + \frac{1}{2}x\right)^6}{2 + 3x}$ in ascending powers of x up to the term in x^3
100. (a) A geometric progression has non-zero first term a and the common ratio r , where $0 < r < 1$. Given that the sum of the first 8 is equal to half the sum to infinity, find the value of r .
- (b) If the 17th term of the progression is 10 find a .
- (c) An arithmetic progression has first term a and the common difference 10. The sum of the first n terms of the progression is 10,000. Express a in terms of n , and show that the n th term of the progression is $\frac{10000}{n} + 5(n-1)$
- (d) Given that the n th term is less than 500, show that $n^2 - 101n + 2000 < 0$ and hence find the largest possible value of n .
101. (a) Solve the following logarithm and exponential simultaneous equations
$$\begin{cases} \log_9 xy = 5/2 \\ \log_3 x \log_3 y = 1 \end{cases}$$

(b) Find the value of k if $x^3 + kx^2 + 2kx + 8$ is a perfect cube.

(c) Express $y = \frac{11x+22}{(2x+3)(x+2)(x+3)}$ into partial fractions, hence

find $\int y dx$.

102. (a) If $2x^6 - x^5 - 2x^3 - 2$ is divided by $(x-1)(x+1)(2x-1)$ what is the remainder?

(b) By using a binomial theorem expand $\sqrt{1+2x}$ up to the 7th term.

(c) Approximate the value of $\sqrt[3]{96}$ correct to 4 decimal places.

103. (a) Find the value of k in the equation $x^2 - 2(1+3k)x + 7(3+2k) = 0$ if it has equal roots.

(b) The value of a and b , when the expression $x^5 + ax^3 + 4x^2 + ax + b = 0$ is divided by $(x-1)(x+1)$ and gives a remainder $2x + 3$.

104. Resolve into partial fractions $\frac{x^3 - x^2 - 3x + 5}{(x-1)(x^2 - 1)}$

105. Let $xu = x^2 + 1$ solve $2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0$

ADVANCED MATHEMATICS

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